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EVIDENCE OF A CONSPIRACY AMONG FIXED POINT THEOREMS

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ABSTRACT. Some generalizations of the Banach contraction theorem replace the global hypothesis that the function be a contraction with various local hypotheses. In this paper, we examine a few of these, and show that, in fact, the functions actually satisfied the global hypothesis after a suitable change of metric. Finally, the techniques developed are applied to prove a new fixed point theorem for locally expansive maps.

The Banach contraction theorem has had numerous generalizations, some of which impose only local hypotheses on the function (see [1], [2], and [3]).

For example, one very pretty theorem, originally due to Edelstein [1], is that if (X, d) is a compact and connected metric space and $f: (X, d) \to (X, d)$ is a local contraction, then f has a unique fixed point. (The statement that $f: (X, d) \to (X, d)$ is a local contraction means that for each $x \in X$, there is an open set U containing x and a real number $M < 1$ so that if y and z belong to U, then $d(f(y), f(z)) \leq Md(y, z)$.) We intend to prove, beyond a shadow of a doubt, that this theorem is actually just a special case of the Banach contraction theorem. We submit the following evidence of a conspiracy between these two theorems: If (X, d) is a compact and connected metric space and $f: (X, d) \to (X, d)$ is a local contraction, then it is possible to find a new metric D for X (yielding the same topology) so that $f: (X, D) \to (X, D)$ is an honest-to-goodness contraction.

Another nice theorem of Edelstein's (see [2]) is that if (X, d) is a compact and connected metric space and $f: (X, d) \to (X, d)$ is locally contractive (that is, each point of X belongs to an open set U so that if y and z are distinct points of U, then $d(f(y), f(z)) < d(y, z)$), then f has a unique fixed point. We will show collusion between this theorem and the fact that contractive maps from a compact metric space to itself have unique fixed points, again by proving a remetrization theorem similar to the one above.

Finally, we use the techniques developed to prove a new fixed point theorem for locally expansive maps.

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Local contractions. Before commencing with the promised remetrization theorem, we shall provide a proof of the local contraction fixed point theorem. We do this for completeness (if not compactness) and to build up some notation we will use later.

Theorem 1.0. *Suppose* (X, d) *is a compact and connected metric space, and* $f\colon (X, d) \to (X, d)$ *is a local contraction. Then* f *has a unique fixed point.*

Proof. First, use compactness to find a positive number δ and a number $M < 1$ so that if $d(x, y) < \delta$ then $d(f(x), f(y)) \leq Md(x, y)$. Next, let $V_1, \dots, V_n$ be a fixed finite open cover of X by sets of diameter less than δ, and let x and y be points of X. Then by connectedness, there is a chain of open sets from x to y, chosen from the sets $V_1, \dots, V_n$. So, certainly, $d(x, y) \leq n\delta$. By considering the image of this chain under f, we see that $d(f(x), f(y)) \leq Mn\delta$, and, more generally, that $d(f^k(x), f^k(y)) \leq M^k n\delta$. Now by choosing k so large that $M^k n < 1$, we see that the set $f^k(X)$ has diameter less than δ. So the function f restricted to the set $f^k(X)$, which maps $f^k(X)$ to itself, is a real live contraction, and by the Banach contraction theorem, this restricted function has a unique fixed point. Finally, since any fixed point of f must belong to $f^k(X)$, we see that f has a unique fixed point.

We are now ready to prove our first remetrization theorem.

Theorem 1.1. *If* (X, d) *is a compact and connected metric space, and* $f\colon (X, d) \to (X, d)$ *is a local contraction, then it is possible to find a new metric* D *for* X *(yielding the same topology) so that* $f\colon (X, D) \to (X, D)$ *is a contraction.*

Proof. We use the notation of the proof of Theorem 1.0. Motivated by the usual proof of the Banach contraction theorem, we define, for $x, y \in X$,

$$D(x, y) = d(x, y) + d(f(x), f(y)) + d(f^2(x), f^2(y)) + \cdots .$$

First of all, this series converges, as can be seen by comparing it with the geometric series $n\delta + Mn\delta + M^2 n\delta + \cdots$. It is easy to check that D is a metric.

To show that D induces the same topology, we will show that the identity function from (X, d) to (X, D) is a homeomorphism. Since (X, d) is compact, showing continuity suffices. So suppose $\epsilon > 0$ is given. Choose $\eta = \min\{\delta, (1 - M)\epsilon\}$. Then if $d(x, y) < \eta$, then $d(x, y) < \delta$, so

$$\begin{aligned} D(x, y) &= d(x, y) + d(f(x), f(y)) + d(f^2(x), f^2(y)) + \cdots \\ &\leq d(x, y) + Md(x, y) + M^2 d(x, y) + \cdots \\ &= (1 - M)^{-1} d(x, y) < \epsilon. \end{aligned}$$

Now we make two observations. First, because $D(x, y) = d(x, y) + D(f(x), f(y))$, $f\colon (X, D) \to (X, D)$ is contractive (that is, if $x \neq y$, then $D(f(x), f(y)) < D(x, y)$). And second, $f\colon (X, D) \to (X, D)$ is still a local contraction—the same δ and M work! So we will complete the proof of Theorem 1.1 when we prove

Lemma 1.2. *If (X, D) is a compact metric space and $f\colon (X, D) \to (X, D)$ is both a local contraction and contractive, then it is a contraction.*

Proof. Suppose that f is not a contraction. Then for each positive integer N, there exist $x_N, y_N \in X$ so that

$$(*) \qquad D(f(x_N), f(y_N)) > (1 - 1/N)D(x_N, y_N).$$

By compactness, and by taking convergent subsequences if necessary, we may assume that $x_N \to a$ and $y_N \to b$. We have two cases: (1) $a \neq b$ and (2) $a = b$. The idea here is that Case (1) violates f being contractive, and Case (2) violates f being a local contraction.

Case (1). Using continuity and taking $x_N \to a$ and $y_N \to b$ in $(*)$, we get that $D(f(a), f(b)) \geq D(a, b)$, which is a contradiction since $a \neq b$ and f is contractive.

Case (2). Since f is a local contraction, there is an open set U containing the point a $(= b)$ and there is an $M < 1$ so that if x and y belong to U, then $D(f(x), f(y)) \leq MD(x, y)$. But since $x_N \to a$ and $y_N \to a$, then for all N sufficiently large, x_N and y_N belong to U. Hence,

$$(1 - 1/N)D(x_N, y_N) < D(f(x_N), f(y_N)) \leq MD(x_N, y_N).$$

Since $x_N \neq y_N$, this implies that for each N, $M > (1 - 1/N)$, i.e. $M \geq 1$, a contradiction.

This completes the proof of Lemma 1.2 and Theorem 1.1.

Locally contractive maps. For completeness again, we begin this section with a short proof of the fixed point theorem for contractive maps.

Theorem 2.0. *Suppose (X, d) is a compact metric space and $f\colon (X, d) \to (X, d)$ is contractive. Then f has a unique fixed point.*

Proof. Define $g\colon X \to \mathbf{R}$ by $g(x) = d(x, f(x))$. By compactness, this function attains a minimum, say at x_0. The point x_0 must be a fixed point, otherwise g attains a smaller value at $f(x_0)$, and uniqueness is obvious.

Now we are ready for our second remetrization theorem.

Theorem 2.1. *Suppose (X, d) is a compact and connected metric space and $f\colon (X, d) \to (X, d)$ is locally contractive. Then there is a new metric D for X (yielding the same topology) so that $f\colon (X, D) \to (X, D)$ is contractive.*

Proof. First, by connectedness, for each positive number ϵ and each pair of points p and q in X, there is a finite set of points $x_0, x_1, x_2, \ldots, x_n$ such that $x_0 = p$, $x_n = q$, and, for $j = 0, 1, 2, \ldots, n-1$, $d(x_j, x_{j+1}) \leq \epsilon$. We will refer to such a set $x_0, x_1, x_2, \ldots, x_n$ as an ϵ-chain of points from p to q.

Next, use compactness to find a positive number δ so that if $x \neq y$ and $d(x, y) < \delta$, then $d(f(x), f(y)) < d(x, y)$.

Now for each $p, q \in X$, define

$$(\dagger) \qquad D(p, q) = \operatorname{glb}\left\{ \sum_{j=0}^{n-1} d(x_j, x_{j+1}) \,\middle|\, x_0, \ldots, x_n \text{ is a } \delta/2\text{-chain of points from } p \text{ to } q \right\}.$$

It is not difficult to show that D is a metric.

To show that D induces the same topology, notice that if $d(p, q) \leq \delta/2$, then the number $d(p, q)$ belongs to the set we are taking the greatest lower bound of, and by the triangle inequality, $d(p, q)$ is a lower bound. Thus, if $d(p, q) \leq \delta/2$, then $D(p, q) = d(p, q)$. Therefore, the identity function from (X, d) to (X, D) is continuous (in fact, a local isometry), and, by compactness of (X, d), a homeomorphism. And so D induces the same topology.

Finally, we must show that if $p \neq q$, then $D(f(p), f(q)) < D(p, q)$. This is almost too easy. For if $x_0, x_1, x_2, \ldots, x_n$ is a $\delta/2$-chain of points from p to q, then $f(x_0), f(x_1), f(x_2), \ldots, f(x_n)$ is a $\delta/2$-chain of points from $f(p)$ to $f(q)$, and $\sum_{j=0}^{n-1} d(f(x_j), f(x_{j+1}))$ is less than $\sum_{j=0}^{n-1} d(x_j, x_{j+1})$ if $p \neq q$. Unfortunately, by taking glb's, we may lose the strict inequality, though this does prove that $D(f(p), f(q)) \leq D(p, q)$. So we need to be a bit more delicate here.

By considering a fixed finite open cover by sets of diameter less than $\delta/2$, the triangle inequality implies that if 3 points in a $\delta/2$-chain of points from p to q lie in the same element of this cover, then at least one of the 3 is superfluous. Thus, we may assume without loss of generality that in calculating $D(p, q)$, we may use a bounded number of x_i's. That is, there is a positive integer N so that

$$D(p, q) = \operatorname{glb}\left\{ \sum_{j=0}^{N-1} d(x_j, x_{j+1}) \,\middle|\, x_0 = p,\ x_N = q \text{ and } d(x_j, x_{j+1}) \leq \delta/2 \text{ for } j = 0, 1, 2, \ldots, N-1 \right\}.$$

Now by using compactness again and by taking convergent subsequences at most $N-1$ times, we can find $x_0, x_1, \ldots, x_N$ such that $x_0 = p$, $x_N = q$, $d(x_j, x_{j+1}) \leq \delta/2$ for $j = 0, 1, \ldots, N-1$, and

$$D(p, q) = d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{N-1}, x_N).$$

Then

$$\begin{aligned} D(f(p), f(q)) &\leq d(f(x_0), f(x_1)) + \cdots + d(f(x_{N-1}), f(x_N)) \\ &< D(p, q) \quad \text{if } p \neq q. \end{aligned}$$

This completes the proof of Theorem 2.1. Notice that this reproves Theorem 1.1 in spades—the same M works!

Locally expansive maps. The statement that $f: (X, d) \rightarrow (X, d)$ is a locally expansive map means that f is continuous and each $x \in X$ belongs to an open set U such that if y and z are distinct points of U, then $d(f(y), f(z)) > d(y, z)$. We call a map $f: X \rightarrow Y$ an open map provided U open in X implies that $f(U)$ is open in Y. What we intend now is to use the techniques of the preceding section to prove a new fixed point theorem for open locally expansive maps. See [3] for the corresponding result for local expansions and for applications. That paper also includes an example showing that the hypothesis of openness cannot be omitted.

Theorem 3.0. *Suppose (X, d) is a compact and connected metric space and that $f: (X, d) \rightarrow (X, d)$ is an open locally expansive map. Then f has a fixed point.*

Proof. Using the fact that f is an open locally expansive map on a compact space, and the technique of [3], we find that there is a positive number δ so that:

(a) if $0 < d(x, y) < \delta$, then $d(f(x), f(y)) > d(x, y)$;

(b) for each x and y such that $d(y, f(x)) < \delta$, there is a unique point p such that $d(x, p) < \delta$ and $y = f(p)$.

Now using connectedness and this δ, define a new metric D for X as in the proof of Theorem 2.1. As we already know, D is a metric for X inducing the same topology.

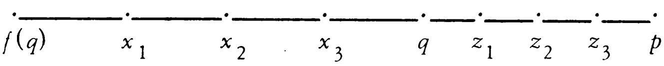
Next, define $g: (X, D) \rightarrow \mathbf{R}$ by $g(x) = D(x, f(x))$. The function g is continuous, and since (X, D) is compact, there is a point $q \in X$ at which g is a minimum. We will show that q is a fixed point for f. Suppose $f(q) \neq q$. In the course of proving Theorem 2.1, we proved that there is a $\delta/2$-chain of points, $x_0, x_1, \ldots, x_N$, from $f(q)$ to q such that

$$D(f(q), q) = \sum_{j=0}^{N-1} d(x_j, x_{j+1}).$$

Without loss of generality, we may assume that the x_j's are all distinct.

We now use properties (a) and (b). Since $d(x_1, f(q)) \leq \delta/2$, there is a unique point z_1 such that $d(q, z_1) \leq \delta/2$, $x_1 = f(z_1)$, and $d(f(q), f(z_1)) > d(q, z_1)$. Similarly, since $d(x_2, f(z_1)) \leq \delta/2$, there is a unique point z_2 such that $d(z_1, z_2) \leq \delta/2$, $x_2 = f(z_2)$, and $d(f(z_1), f(z_2)) > d(z_1, z_2)$.

Continuing in this manner we construct a $\delta/2$-chain of points, $z_0, z_1, \ldots, z_N$, from q to a point p such that $f(p) = q$ and, for each j, $d(x_j, x_{j+1}) > d(z_j, z_{j+1})$:



But then

$$D(p, f(p)) \leq \sum_{j=0}^{N-1} d(z_j, z_{j+1}) < \sum_{j=0}^{N-1} d(x_j, x_{j+1}) = D(q, f(q)),$$

contradicting q being a point at which g is a minimum.

This completes the proof of Theorem 3.0.

The prosecution rests.

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