PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 53, Number 1, November 1975

STABILITY THEORY FOR HILL EQUATIONS WITH GENERALIZED COEFFICIENT

H. GUGGENHEIMER[1]

ABSTRACT. A simple geometric proof is given for the existence of Ljapunov's intervals of stability and instability for Hill equations with generalized coefficients.

The basic theorem of Ljapunov on the stability of Hill equations

$$x'' + \lambda p(t)x = 0, \quad p(t+T) = p(t), \quad p' \tag{1}$$

is:

(L) There exists a sequence of eigenvalues

$$\lambda''_{-1} = 0 < \lambda'_0 \leq \lambda''_0 < \cdots < \lambda'_k \leq \lambda''_k < \cdots$$

of (1) for periodic boundary value problems

$$x(t+T) = (-1)^{k+1}x(t), \quad x'(t+T) = (-1)^{k+1}x'(t).$$

The intervals of stability of (1) are $(\lambda''_k, \lambda'_{k+1})$ and the intervals of instability $[\lambda'_k, \lambda''_k]$ (except if $\lambda'_k = \lambda''_k$ when all solutions of $x + \lambda'_k px = 0$ are periodic).

A complete proof of the corresponding theorem for

$$x'' + (\lambda - q(t))x = 0 \tag{2}$$

and continuous $q(t)$ is given by Magnus and Winkler [4]. In the case of (2), no hypothesis is needed on the sign of q. The proof of [4] is laborious; it is classified as "very complicated" in [3]. In the present note, we give a proof of (L), based on elementary geometry, Sturm-Liouville and Floquet theories, that is short and easy for p or q continuous functions. With the few additional arguments presented here, the proof is valid if

(A) $p(t)$ is periodic of period T,

(B) $p(t)$ is the generalized derivative of a bounded, monotone increasing, one-sided continuous function and there exists $\epsilon > 0$ such that

$$\int_{t_0}^{t_0+\Delta} p(t)\,dt > \epsilon\Delta \tag{B$_1$}$$

for all t_0 and Δ.

Received by the editors September 6, 1974.
AMS (MOS) subject classifications (1970). Primary 34C10; Secondary 34C25.
[1] Research partially supported by NSF Grant GP-27960.

The monotonicity condition would be superfluous for (2). If (B) holds, (1) (or (2)) is equivalent [1, (2)] to a linear system with continuous coefficients and, therefore, the initial value problem has unique solutions. We denote by $x^u(t)$ the curve $(x_1^u(t), x_2^u(t))$ whose coordinate functions are solutions of (1) for the standard initial conditions $x^u(u) = (1, 0)$, $x^{u\prime}(u) = (0, 1)$. The curve $x^u(t)$ is locally convex, its angular velocity is positive, and its Wronskian W is unity. The solutions of (1) have forward and backward derivatives. If $p(t)$ is the generalized derivative of $Q(t)$, then [1, (9)]

$$x'_+(t) = x'_-(t) - \Delta Q(t)x(t) \tag{3}$$

where $\Delta Q(t) = Q(t+0) - Q(t-0)$. Every scalar solution is differentiable at its zeros. The conjugate point $c(u, \lambda)$ is the smallest value $t > u$ for which $x^u(t)$ is linearly dependent on $x^u(u)$. The kth conjugate point is $c_k(u, \lambda) = c(c_{k-1}(u, \lambda), \lambda)$ where $c = c_0$. The conjugate points are the zeros of x_2^u and for the angle

$$\alpha(t) = \tan^{-1} x_2^u(t)/x_1^u(t)$$

we have

$$\alpha(c_k(u, \lambda)) = \alpha(u) + (k+1)\pi. \tag{4}$$

The function $c_k(u, \lambda)$ is differentiable in both variables [2, pp. 225–226], and for $x_i(u) = 0$ we have [2, (8)]

$$\partial c/\partial u = x_i'(u)^2/x_i'(c(u, \lambda))^2. \tag{5}$$

The independent variable t is two times the area covered by the vector $x(t)$ (up to an additive constant).

Lemma 1. *The function $\alpha'(t)$ is strictly increasing with λ, $c(u, \lambda)$ is strictly decreasing with λ.*

The lemma is simply Sturm's comparison theorem which is valid in our case [2, Theorem 1]. In fact, $\alpha' = (x_1^2 + x_2^2)^{-1}$, and for identical initial conditions the comparison theorem implies that for larger λ the radius $|x|$ is smaller and, since t is the area, the angular velocity must be greater. The second assertion is an immediate consequence of the first.

Lemma 2. *The problem $x'' + \lambda p(t)x = 0$, $x(u) = x(u+T) = 0$, has a discrete spectrum of eigenvalues $\lambda_k(u)$ of all orders, $\lambda_k(u) \to \infty$.*

The index of the eigenvalue is the number of zeros of an eigenfunction for $u \leq t < u + T$. For the vector x^u,

$$\alpha(u+T) = (k+1)\pi. \tag{6}$$

By Lemma 1, there can be no more than one eigenvalue to an index. By the

Sturm comparison theorem and (B_1), there exist eigenvalues of all orders. The remainder of the lemma is proved as for continuous coefficients.

Lemma 2 will not hold for equation (1) without condition (B_1). For example, for $u = 0$ and $p(t)$ a sum of δ-functions of mass 1 at $(2n + 1)T/2$, only λ_0 will exist.

The eigenvalues can also be defined by

$$c_k(u, \lambda_k(u)) = u + T \tag{7}$$

and, hence, are differentiable functions of u [2, (10)].

Definition [5]. $\lambda_k' = \min \lambda_k(u)$, $\lambda_k'' = \max \lambda_k(u)$. It is sufficient to consider an interval $0 \leq u < T$. By (7),

$$\frac{\partial c_k}{\partial u} + \frac{\partial c_k}{\partial \lambda}\frac{d\lambda_k}{du} = 1$$

and for both $x^u(t, \lambda_k')$ and $x^u(t, \lambda_k'')$ we have

$$x_2^u(u + T) = (-1)^{k+1}x_2^u(u) = 0, \quad x_2^{u\prime}(u + T) = (-1)^{k+1}x_2^{u\prime}(u).$$

The sign follows from (6) and $W = 1$.

We now use Floquet theory (e.g. [4, §5.2]). If $\lambda(u)$ is a solution of a problem of Lemma 2, it follows from (6) that the matrix that maps $(x^u(u), x^{u\prime}(u))$ onto $(x^u(u + T), x^{u\prime}(u + T))$ has a real eigenvalue and $\lambda(u)$ is unstable for (1). If there is not an eigenvalue for any u and k, no such matrix will have a real eigenvalue and λ is stable. Hence, all we have to prove is that the λ_k' and λ_k'' interlace in the right order.

Lemma 3. $\lambda_k(u) > \lambda_{k-1}(v)$ *for all* u *and* v.

The arc $x^u(t)$ intersects the x_1-axis at $c_i(u, \lambda_k(u))$, $i = 0, 1, \ldots$. We estimate $c_{k-1}(v, \lambda_k(u))$. Since everything is periodic, we can assume $u \leq v < u + T$; more specifically, we assume $c_{j-1}(u, \lambda_k(u)) \leq v < c_j(u, \lambda_k(u))$ and put $u = c_{-1}(u, \lambda_k(u))$. The arc x^u intersects the line $Ox^u(v)$ once between two consecutive conjugate points. Hence,

$$c_{j+k-1}(u, \lambda_k(u)) \leq c_{k-1}(v, \lambda_k(u)) < c_{j+k}(u, \lambda_k(u)).$$

Since $c_{j+k}(u, \lambda_k(u)) = c_{j-2}(u + T, \lambda_k(u))$, we have

$$\begin{aligned} c_{k-1}(v, \lambda_k(u)) - v &< c_{j+k}(u, \lambda_k(u)) - c_{j-1}(u, \lambda_k(u)) \\ &= [c_{j+k}(u, \lambda_k(u)) - (u + T)] + [u + T - c_{j-1}(u, \lambda_k(u))] \\ &= u + T - c_{j-1}(u, \lambda_k(u)) + c_{j-2}(u, \lambda_k(u)) - u \\ &< T = c_k(u, \lambda_k(u)) - u. \end{aligned}$$

Since $c_{k-1}(v, \lambda_{k-1}(v)) - v = T$, Lemma 3 follows from Lemma 1. We have proved

Theorem. (L) *holds under the hypotheses* (A), (B).

REFERENCES

1. H. W. Guggenheimer, *Geometric theory of differential equations.* I. *Second order linear equations*, SIAM J. Math. Anal. **2** (1971), 233–241. MR **44** #2977.

2. ———, *Geometric theory of differential equations.* III. *Second order equations on the reals*, Arch. Rational Mech. Anal. **41** (1971), 219–240.

3. H. Hochstadt, *Differential equations: A modern approach*, Holt, Rinehart and Winston, New York, 1964. MR **35** #433.

4. W. Magnus and S. Winkler, *Hill's equation*, Interscience Tracts in Pure and Appl. Math., no. 20, Interscience, New York, 1966. MR **33** #5991.

5. V. M. Staržinskiĭ, *A survey of works on conditions of stability of the trivial solution of a system of linear differential equations with periodic coefficients*, Prikl. Mat. Meh. **18** (1954), 469–510; English transl., Amer. Math. Soc. Transl. (2) **1** (1955), 189–237. MR **16**, 249; **17**, 484.

DEPARTMENT OF MATHEMATICS, POLYTECHNIC INSTITUTE OF NEW YORK, BROOKLYN, NEW YORK 11201