A CLOSED GRAPH THEOREM

S. O. IYAHEN

ABSTRACT. A closed graph theorem is proved, implying that a Hausdorff locally convex space $E$ need not be barrelled if every closed linear map from $E$ into $F$ is continuous, where $F$ is a reflexive Fréchet or LF-space or a space of distributions.

A Hausdorff locally convex space $(E, \tau)$ is barrelled if and only if for each Banach space $(F, \xi)$, a closed linear map from $(E, \tau)$ into $(F, \xi)$ is necessarily continuous. This result of M. Mahowald [2, Theorem 2.2] is shown here (Corollary) to be false if instead $(F, \xi)$ is a reflexive Fréchet or LF-space.

Theorem. Let $(E, \tau)$ be a Hausdorff locally convex space. Suppose that for a family $(S_a)_{a \in \Phi}$ of absolutely convex compact subsets, there is no locally convex topology on $E$ strictly finer than $\tau$ which coincides with $\tau$ on each $S_a$. Let $(F, \xi)$ be a Hausdorff locally convex space in which every closed bounded set is compact. If every closed linear map from any Banach space into $(F, \xi)$ is bounded, then a closed linear map $f$ from $(E, \tau)$ into $(F, \xi)$ is necessarily continuous.

Proof. Let $E_a$ denote the linear span of $S_a$, and $\tau_a$ the $\tau$-induced topology on $E_a$. Let $\eta_a$ denote the Banach space topology on $E_a$ with the sequence $(S_a/n)_{n=1}^{\infty}$ of sets as a base of neighborhoods of the origin.

As the graph of the restriction map of $f$ to $E_a$ is closed in $(E_a, \tau_a) \times (F, \xi)$, it is necessarily closed in $(E_a, \eta_a) \times (F, \xi)$ since $\eta_a$ is finer than $\tau_a$. By the hypothesis then, the closure $T_a$ of $f(S_a)$ in $(F, \xi)$ is compact. The graph $G_a$ of the restriction map $f_a$ of $f$ to $S_a$ is closed in the compact Hausdorff space $X_a = S_a \times T_a$, where $S_a$, $T_a$ are considered under the $\tau$- and $\xi$-induced topologies, respectively.

Let $P_{S_a}$, $P_{T_a}$ be the projection maps of $X_a$ onto $S_a$, $T_a$ respectively, and $P_{G_a}$ the restriction of $P_{S_a}$ to $G_a$. As $P_{G_a}$ is continuous and one-to-one, it is necessarily a homeomorphism, since $G$ is compact. Therefore each map $f_a = P_{T_a} \circ P_{G_a}^{-1}: S_a \rightarrow (F, \xi)$ is continuous. This implies that $f: (E, \tau) \rightarrow (F, \xi)$ is continuous.

In the Theorem, $(E, \tau)$ could be the dual of any complete Hausdorff locally convex space under the topology of compact convergence [1, p. 16,
Example B], and \((F, \xi)\) could be any Montel Fréchet or LF-space or any of the distribution spaces \(\mathcal{E}, \mathcal{E}', \mathcal{D}, \mathcal{D}', \mathcal{J}, \mathcal{J}', \mathcal{O}_M, \mathcal{O}_M', \mathcal{O}_c, \mathcal{O}_c'\) by [3].

**Corollary.** Let \((E, \tau)\) be the dual of a complete Hausdorff locally convex space under the Mackey topology \(\tau\). If \((F, \xi)\) is a reflexive Fréchet or LF-space or any of the distribution spaces above, then any closed linear map \(f\) from \((E, \tau)\) into \((F, \xi)\) is necessarily continuous.

**Proof.** Let \(\tau_1\) denote the topology on \(E\) of compact convergence and \(\xi_0\) the weak topology on \(F\) associated with \(\xi\). The graph of \(f\) is necessarily closed in \((E, \tau_1) \times (F, \xi_0)\), and by the Theorem then \(f: (E, \tau_1) \to (F, \xi_0)\) is continuous. This implies that \(f: (E, \tau) \to (F, \xi)\) is continuous.

**REFERENCES**