SOME COMMUTATIVITY RESULTS FOR RINGS WITH TWO-VARIABLE CONSTRAINTS

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ABSTRACT. It is proved that an associative ring $R$ has nil commutator ideal if for each $x, y \in R$, there is a polynomial $p(X) \in XZ[X]$ for which $xy - yp(x)$ is central. Two restrictions on the $p(X)$ which guarantee commutativity are established.

Let $P$ denote the set of those polynomials in two noncommuting indeterminates which have integer coefficients and constant term zero. We consider associative rings $R$ with the property that for each ordered pair $(x, y)$ of elements of $R$, there exists a polynomial $p(X, Y) \in P$, depending on $(x, y)$, for which

\[ xy - p(x, y) \in Z, \]

where $Z$ denotes the center of $R$.

Putcha and Yaqub [6] have shown that if each $p(X, Y)$ in (1) is a sum of terms each of degree at least two in both $X$ and $Y$, then $R \subseteq Z$, and hence, by a long-standing theorem of Herstein [4], $R$ has nil commutator ideal. Unless the $p(X, Y)$ in (1) are restricted in some fashion, $R$ may be badly noncommutative—indeed the ring of $2 \times 2$ matrices over $GF(2)$ satisfies a condition of type (1), obtained by linearizing the identity $x^2 = x^8$. However, less severe restrictions than those imposed by Putcha and Yaqub, while not implying that any power of $R$ is central, will still yield the result that $R$ has nil commutator ideal; and this note deals with one such condition, together with some special cases of it which actually yield commutativity.

Letting $XZ[X]$ denote the ring of polynomials over the integers which have zero constant term, we state our major theorem as follows:

**Theorem 1.** Let $R$ be a ring such that for each ordered pair $(x, y)$ of elements of $R$ there exists a polynomial $p(X) \in XZ[X]$, depending on $(x, y)$, for which

\[ xy - yp(x) \in Z. \]

Then the commutator ideal $C(R)$ is nil and the nilpotent elements of $R$ form an ideal.

1. Proof of Theorem 1.

**Lemma 1.** Let $R$ be a ring satisfying an identity $q(X) = 0$, where $q(X)$ is a polynomial in a finite number of noncommuting indeterminates, its coe-
ficients being integers with highest common factor 1. If there exists no prime p for which the ring of $2 \times 2$ matrices over $GF(p)$ satisfies $q(X) = 0$, then $R$ has nil commutator ideal and the nilpotent elements of $R$ form an ideal.

The proof of this lemma, which depends on a deep result of Amitsur on PI-rings, may be found in [2].

**Lemma 2.** Let $R$ be a ring satisfying the hypothesis of Theorem 1 and having no nonzero divisors of zero; and let $(x, y)$ be an arbitrary ordered pair of elements of $R$. If $p(X) \in XZ[X]$ is such that $xy - yp(x) \in Z$, then $xy = y^2x$ or $xy = yp(x)$.

**Proof.** Suppose that $xy^2 \neq y^2x$, and write

$$xy = yp(x) + z,$$

where $z \in Z$; and let $p_1(X) \in XZ[X]$ be such that

$$x^2y - yp_1(x^2) \in Z.$$

Repeated substitution of (3) in (4) yields $x(yp(x) + z) - yp_1(x^2) \in Z$,

$$(yp(x) + z)p(x) + xz - yp_1(x^2) \in Z,$$

and finally

$$y((p(x))^2 - p_1(x^2)) + z(x + p(x)) \in Z.$$

If $(p(x))^2 - p_1(x^2) \neq 0$, (5) implies that $xy = yx$, contrary to our supposition that $xy^2 \neq y^2x$; hence

$$(p(x))^2 - p_1(x^2) = 0 \quad \text{and} \quad z(p(x) + x) \in Z,$$

so that $z = 0$ or $p(x) + x \in Z$. But if $p(x) + x \in Z$, then (3) yields $xy - yp(x) = xy - y(p(x) + x) + yx \in Z$, implying that $y$ commutes with $xy + yx$ and, hence, that $y^2$ commutes with $x$; therefore $z = 0$ and (3) now shows that $xy = yp(x)$.

**Proof of Theorem 1.** It will suffice to show that prime rings satisfying the hypothesis of Theorem 1 are commutative (see [2]). Accordingly, let $R$ be such a prime ring; we first show that $R$ has no nonzero divisors of zero. Suppose that $ab = 0$, $a \neq 0$, and $r$ is an arbitrary element of $R$. There exists $q(X) \in XZ[X]$ for which $b(ra) - (ra)q(b) \in Z$; and since $aq(b) = 0$, we have $b(ra) \in Z$ and thus $sa(bra) = 0 = (bra)s$ for all $s \in R$. The primeness of $R$ now implies that $bra = 0$ and hence that $b = 0$.

Assume that $R$ is a noncommutative prime ring satisfying (2). The identity

$$(xy^2 - y^2x)(yx^2 - x^2y)(xy^2x - y^2xy) = 0$$

is not satisfied by the ring of $2 \times 2$ matrices over any field $GF(p)$, as may be verified by substituting the matrices $[1 1, 0 0]$ and $[1 0, 0 0]$ for $x$ and $y$ respectively; thus, by Lemma 1, $R$ cannot satisfy (7), and there must exist elements $a, b$ of $R$ for which $ab^2 - b^2a$, $ba^2 - a^2b$, and $ab^2a - ba^2b$ are all nonzero;
If \( p(X) \in X[Z[X]] \) is such that \( ab - bp(a) \in Z \), it follows from Lemma 2 that
\[
(8) \quad ab = bp(a).
\]
Now let \( s(X) \in X[Z[X]] \) satisfy
\[
(9) \quad bp(a) - p(a)s(b) \in Z
\]
and apply the result of Lemma 2 to the ordered pair \( (b, p(a)) \). If \((p(a))^2 b = b(p(a))^2\), it follows from (8) that \( a^2 b = a(bp(a)) = b(p(a))^2 = (p(a))^2 b \), so that \((p(a))^2 = a^2 \) commutes with \( b \), contrary to the choice of \( a \) and \( b \).
Therefore, by Lemma 2, \( bp(a) = p(a)s(b) \), which combines with (8) to give
\[
(10) \quad ab = p(a)s(b).
\]
Now it is immediate from Lemma 2 that \( R \) is an Ore domain and can be embedded in a division ring \( D \). In \( D \), (10) implies that \( b(s(b))^{-1} = a^{-1} p(a) \) commutes with both \( a \) and \( b \); and (8) written in the form \( ab = baa^{-1} p(a) \) shows that \( ab \) and \( ba \) commute, contrary to the original choice of \( a \) and \( b \).
This contradiction completes the proof of Theorem 1.

2. Two commutativity theorems. In this section we single out two conditions of type (2) which imply commutativity.

**Theorem 2.** Let \( R \) be a ring such that for every ordered pair \((x, y)\) of elements of \( R \), there exists an integer \( n = n(x, y) > 1 \) for which \( xy = yx^n \).

Then \( R \) is commutative.

**Lemma 3.** Any ring \( R \) satisfying the hypothesis of Theorem 2 has each of the following properties:

(a) Idempotents of \( R \) are central.

(b) \( R \) is a duo ring (i.e. one-sided ideals are two-sided); moreover \( ab = 0 \) implies \( ba = 0 \), so that there is no distinction between right and left zero divisors.

(c) Commutators in \( R \) are central.

(d) If \( a, b \in R \) are such that \( a(ab - ba) = b(ab - ba) = 0 \), then \( ab - ba = 0 \); similarly, if \( a(ab - ba)x = b(ab - ba)x = 0 \) for some \( x \in R \), then \( (ab - ba)x = 0 \).

**Proof.** (a) If \( x \in R \) and \( e \) is idempotent, there exist positive integers \( m, n \) such that \( e(ex - exe) = (ex - exe)e^m \) and \( e(xe - exe) = (xe - exe)e^n \); hence \( ex - exe = xe - exe = 0 \).

(b) Let \( I \) be a right ideal of \( R \), \( a \in I \) and \( r \in R \); note that since \( ra = ar^n \) for some \( n \geq 1 \), \( ra \in I \). Thus all right ideals are two-sided, and a similar argument holds for left ideals.

Now let \( ab = 0 \). Since \( ba = ab^n \) for some \( n \geq 1 \), \( ba = 0 \) as well.

(c) By Theorem 1 the commutator ideal is nil and, hence, contained in the Jacobson radical \( J(R) \); therefore, it will suffice to show \( J(R) \subseteq Z \). If we
assume the existence of an element \( a \in J(R) \setminus \mathbb{Z} \), then there is an element \( b \in \mathbb{R} \) and integers \( m, n > 1 \) for which \( ab = ba^m_n \) and \( ba = ab_n^m \neq ab \). It follows that \( ab = ahh^n-1a^{m-1}_n \); and because \( h^n-1a^{m-1}_n \in J(R) \), we now have \( ab = 0 \). Similarly, \( ba = 0 \) and we have a contradiction.

(d) Suppose \( a(ab - ba) = b(ab - ba) = 0 \); in view of (c), \( a^2b = ba^2 \) and \( b^2a = ab^2 \). Suppose \( ab - ba \neq 0 \) and let \( m, n > 1 \) be such that \( ab = ba^m \) and \( ba = ab_n^m \). Substituting each of these expressions into the other yields \( ab = ab^n \cdot a^{m-1}_n \) and \( ba = ba^m \cdot b^{n-1}_n \). If \( m \) and \( n \) are both even we thus get \( ab = ba = a^m \cdot b^n \); on the other hand, if one of \( n, m \) is odd, we have

\[
ab - ba = ab^n \cdot a^{m-1}_n - ba^m \cdot b^{n-1}_n = (ab - ba)a^m \cdot b^n - 1,
\]

which is zero since \( (ab - ba)a = 0 \).

Finally, if \( x \in \mathbb{R} \) and \( A \) is the annihilator of \( x \), we get the second statement of (d) by applying the preceding argument to the ring \( \mathbb{R}/A \).

**Proof of Theorem 2.** It will suffice to prove commutativity under the additional hypothesis that \( R \) is subdirectly irreducible, in which case (since \( R \) is a duo ring) the set of zero divisors is precisely the annihilator of the unique minimal ideal \( S \) [1, Lemma 3].

The initial step is to show that zero divisors in \( R \) are central. Accordingly, suppose \( a \) is a noncentral zero divisor which fails to commute with some element \( b \in \mathbb{R} \); and consider the case where \( b \) is also a zero divisor. Then by (d) of Lemma 2, we have one of \( (ab - ba)a \) and \( (ab - ba)b \) different from 0 and \( (ab - ba)R \) is a nontrivial ideal; therefore if \( 0 \neq s \in S \), there exists an element \( x \in \mathbb{R} \) for which \( s = (ab - ba)x \). But \( 0 = as = bs = a(ab - ba)x = b(ab - ba)x \), and from (d) of Lemma 2 we then get \( (ab - ba)x = 0 \), a contradiction. Now consider the case where \( b \) is not a zero divisor and let \( m, n > 1 \) be such that \( ab = ba^m_n \) and \( ba = ab_n^m \). Since \( ab \) is a zero divisor, \( ab \) and \( a \) commute, so that \( a(ab - ba) = (ab - ba)a = 0 \) and \( a^2 \) commutes with \( b \). If \( m \) is odd, repeating some of the computation in Lemma 2(d) shows that \( ab - ba = (ab - ba)a^m - 1b^{n-1}_n = 0 \); on the other hand, if \( m \) is even, \( ab = a^m \cdot b, a^m = a \), and \( a_n^m - 1 \) is a nonzero idempotent. Recalling that any nonzero central idempotent of a subdirectly irreducible ring must be a multiplicative identity element, we get a contradiction of the fact that \( a \) was a zero divisor. Therefore zero divisors of \( R \) are central.

Now suppose that \( R \) is not commutative and \( b \notin \mathbb{Z} \). There then exist \( a \in \mathbb{R} \) not commuting with \( b \) and an integer \( j > 1 \) such that \( ba = ab^j \). Since \( a \) cannot be a zero divisor and since \( ab - ba = a(b - b^j) \) is a zero divisor (nilpotent, in fact), \( b - b^j \) must be a zero divisor, hence central. We have now arrived at a contradiction of Herstein's well-known result that a ring \( R \) is commutative if for each \( x \in \mathbb{R} \), there is an integer \( n(x) > 1 \) for which \( x - x^\infty \in \mathbb{Z} \); and our proof is complete.

**Theorem 3.** Let \( R \) be a ring such that for every ordered pair \( (x, y) \) of
elements of $R$, there is a polynomial $p(X) \in \mathbb{Z}[X]$ such that $xy = yxp(x)$. Then $R$ is commutative.

**Proof.** Again applying the given condition to $e$, $ex - exe$, and $xe - exe$ shows that idempotents must be central. Also, since $x^2 = x^2p(x)$ for some $p(X) \in \mathbb{Z}[X]$, $R$ is periodic by a result of Chacron [3]; therefore, $R$ is either nil or contains a nonzero idempotent.

Suppose now that $R$ is subdirectly irreducible. If $R$ contains a nonzero idempotent, then it must have an identity; thus, for each $x \in R$ we have $x = xp(x)$, and $R$ is commutative by the major theorem of [5]. On the other hand, if $R$ is nil we have

$$xy = yxp(x) = xyq(y)p(x) = yxp(x)q(y)p(x) = yxq(y)r(x)$$

for an appropriate element $r(X) \in \mathbb{Z}[X]$. In particular, $xy = yxyz_1$ for some element $z_1 \in R$; and, continuing inductively, for each positive integer $n$ we get an element $z_n \in R$ for which $xy = y^nxyz_n$, so that $xy = 0$ and $R$ is a zero ring. Therefore, if $R$ is subdirectly irreducible, it is commutative; and the proof of Theorem 3 is finished.

The hypothesis of Theorem 3 cannot be weakened to the condition that $xy - yxp(x) \in \mathbb{Z}$, as we see by noting that there exist noncommutative rings satisfying the identity $x^2 = 0$. However, it may be of some interest (but not enough to justify including the proof) to note that rings satisfying the weaker hypothesis are polynomial-identity rings—satisfying the identity $[[x, y], z]z^2[x, y] = 0$.

**REFERENCES**


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