CYCLOTONIC SPLITTING FIELDS FOR GROUP CHARACTERS

MARK BENARD

ABSTRACT. This paper is concerned with cyclotomic splitting fields for a real-valued irreducible character of a finite group. The fields considered are of the form $\mathbb{Q}(\zeta_m)$, where $m$ is either an odd prime or a power of 2.

Let $\chi$ be an irreducible character of $G$ and let $\zeta_m$ be a primitive $m$th root of unity. A famous theorem of Richard Brauer states that if $m$ is the exponent of $G$, then $\mathbb{Q}(\zeta_m)$ is a splitting field for $G$. In a paper where he gives his second proof of this theorem, Brauer states the following proposition without proof [2, Theorem 3]: If $\chi$ is a real-valued character of $G$, then there exists an element of $G$ whose order $m$ is either an odd prime or a power of 2 such that $\mathbb{Q}(\zeta_m)$ splits $\chi$. The examples given below show that this proposition is actually false. One weaker theorem is proved by B. Fein [3]. The Theorem given below is another attempt to substitute for Brauer’s proposition.

Let $k$ be a field of characteristic 0. The pair $(G, \chi)$ is said to be $k$-special if there exists a normal, cyclic, self-centralizing subgroup $A$ of $G$ and a faithful linear character $\lambda$ of $A$ such that $\chi = \lambda^G$ and $G/A$ acts on $\lambda$ as $\text{Gal}(k(\lambda)/k(\chi))$. Many questions on the Schur index reduce to considering such $k$-special pairs. Basic results on the Schur index can be found in Yamada [4].

Theorem. Suppose that $\chi$ is a real-valued character of $G$ and $G$ contains no elements of order $4n$ with $n$ odd and $n > 1$. Then there exists an integer $m$ dividing the exponent of $G$ such that $m$ is either an odd prime or 4, and such that $\mathbb{Q}(\zeta_m)$ splits $\chi$.

Proof. To prove the Theorem, it is necessary to show that the Schur index $m_F(\chi)$ equals 1 for some field $F = \mathbb{Q}(\zeta_m)$ as specified above. Since $\chi$ is real-valued, then $m_\mathbb{Q}(\chi) \leq 2$ by the Brauer-Speiser theorem. By the Brauer-Witt theorem, it suffices to consider $\mathbb{Q}(\chi)$-special pairs $(G, \chi)$ where $G/A$ is a 2-group.

If $G$ is a 2-group, then $m_\mathbb{Q}(\chi) = 1$ if $\exp(G) = 2$. If $4|\exp(G)$, then $m$
= 4 satisfies the conclusion of the Theorem. For the remainder of the proof, assume that there exists an odd prime \( q \) which divides \( |G| \). It will be shown that \( F = Q(\epsilon_q) \) splits \( \chi \).

Assume \( m_{Q_p}(\chi) = 2 \). Let \( p \) be a prime such that \( m_{Q_p}(\chi) = 2 \). Let \( H \) be a subgroup of \( G \) such that \( A \leq H \) and \( H/A \) acts on \( \lambda \) as \( \text{Gal}(Q_p(\lambda)/Q_p(\chi)) \). Set \( \phi = \lambda^H \). Then \( m_{Q_p}(\phi) = m_{Q_p}(\chi) = 2 \). Suppose \( H \) contains no element of order 4. Then a Sylow 2-subgroup of \( G \) is elementary abelian and \( A \) has a complement \( T \) in \( H \). Thus \( \phi(1) = |T|! \) and \( (\phi, (1_\chi)^H) = 1 \), so \( m_{Q_p}(\phi) = 1 \), which is a contradiction. Therefore \( H \) contains an element of order 4. Let \( x \) be an element of order \( q \) in \( A \). Since \( G \) contains no element of order \( 4q \), then \( x \notin \mathbb{Z}(H) \). Since \( \lambda \) is faithful and \( H/A \) acts on \( \lambda \) as \( \text{Gal}(Q_p(\lambda)/Q_p(\chi)) \), then \( 2|Q_p(\chi, \epsilon_q) : Q_p| \). Thus, if \( k = Q_p(\epsilon_q) \), then \( m_k(\chi) = 1 \). Therefore \( m_F(\chi) = 1 \) for \( F = Q(\epsilon_q) \).

Example (1). Define \( G = \langle a, b, c, x, y, w \rangle \) with the following relations:

\[
\begin{align*}
a^5 &= b^{11} = c^{43} = z^2 = x^4 = w^4 = 1, & y^{10} = z, \\
[x, w] &= z, & x^{-1}ax = a^2, & y^{-1}by = b^2, & w^{-1}cw = c^3.
\end{align*}
\]

Then \( \exp(G) = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 43 \). Let \( A = \langle a, b, c, z \rangle \) and let \( \lambda \) be a faithful linear character of \( A \). Then \( A \triangleleft G \) and \( \chi = \lambda^G \) is a rational-valued irreducible character of \( G \). The \( p \)-local Schur indices of \( \chi \) can be calculated by using either the formula of Berman [1, §4] or Yamada [4, Chapter 4]. The index \( m_{Q_p}(\chi) = 2 \) exactly when \( p = 5, 11, 43, \) and \( \infty \). Furthermore, if \( m \in \{4, 3, 5, 7, 11, 43\} \), there exists \( p \in \{5, 11, 43, \infty\} \) such that \( |Q_p(\epsilon_m) : Q_p| \) is odd. Thus \( Q(\epsilon_m) \) fails to split \( \chi \) for each such \( m \). Hence Brauer's proposition is false.

Example (2). Another example shows that if \( \exp(G) \) is replaced by \( |G| \), then the proposition is still false. Define \( G = \langle a, b, c, z, x, y, w \rangle \) with the following relations:

\[
\begin{align*}
a^{17} = b^{31} = c^{103} = z^2 = x^2 = y^2 = 1, & w^2 = z, \\
[x, y] = z, & x^{-1}ax = a^{-1}, & y^{-1}by = b^{-1}, & w^{-1}cw = c^{-1}.
\end{align*}
\]

Then \( |G| = 2^4 \cdot 17 \cdot 31 \cdot 103 \). Set \( A = \langle a, b, c, z \rangle \), \( \lambda \) a faithful character of \( A \), and \( \chi = \lambda^G \). Then \( \chi \) is real-valued and has local Schur index 2 at \( 17, 31, \) and \( 103 \). Furthermore, if \( m \in \{16, 17, 31, 103\} \), there exists \( p \in \{17, 31, 103\} \) such that \( |Q_p(\epsilon_m) : Q_p| \) is odd. Therefore \( Q(\epsilon_m) \) fails to split \( \chi \) for each \( m \).

The following result shows that this situation cannot happen if \( \chi \) is rational-valued.

**Proposition.** Let \( \chi \) be an irreducible character of \( G \) such that \( Q(\chi) \) is an extension of \( Q \) of odd degree. If \( |G| = 2^n, n \) odd, then \( Q(\epsilon_2^c) \) splits \( \chi \).
Proof. By the Brauer-Speiser theorem, \( m_Q(\chi) \leq 2 \). By the Brauer-Witt theorem, it suffices to consider \( Q(\chi) \)-special pairs \((G, \chi)\) where \( G/A \) is a 2-group. Since \( Q(\chi)/Q \) has odd degree, \( G/A \) is isomorphic to a Sylow 2-subgroup of \( \text{Gal}(Q(\lambda)/Q) \).

Suppose \( m_Q(\chi) = 2 \). Then \( \chi \) cannot be linear, so \( G \neq A \). Hence \( 2 \mid |G:A| \). Let \( T \) be a Sylow 2-subgroup of \( G \). If \( A \cap T = \{1\} \), then \( \chi(1) = |G:A| = |T| \) and \( (\chi, (1_T)^G) = 1 \). In that case, \( m_Q(\chi) = 1 \), which is a contradiction. Hence \( 2 \mid |A| \) so \( 4 \mid |G| \) and \( c \geq 2 \). Thus \( 2 \mid |Q_p(\varepsilon_2c):Q_p| \) for \( p = 2, \infty \). In particular, \( Q_2(\varepsilon_2c) \) and \( Q_\infty(\varepsilon_2c) \) each split \( \chi \).

Let \( p \) be an odd prime with \( p - 1 = 2^a b \), \( b \) odd. Suppose \( m_{Q_p}(\chi) = 2 \). Then \( p \mid |A| \). Since \( \lambda \) is faithful and \( G/A \) is isomorphic to a Sylow 2-subgroup of \( \text{Gal}(Q(\lambda)/Q) \), \( 2^a \mid |G:A| \). Therefore, \( c \geq a + 1 \). Hence, \( 2 \mid |Q_p(\varepsilon_2c):Q_p| \) so \( Q_p(\varepsilon_2c) \) splits \( \chi \).

Therefore \( Q(\varepsilon_2c) \) splits \( \chi \).

REFERENCES


DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LOUISIANA 70118