ON A PROBLEM OF J. L. TAYLOR

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ABSTRACT. Let $S$ be the structure semigroup of a measure algebra $M(G)$ and $K$ be the union of all maximal groups of $S$. Taylor proposed the following problem: Are there L.C.A. groups $G$ with nontrivial measures concentrated on $K$? The purpose of this paper is to give a positive solution to this problem.

Let $G$ be a locally compact abelian group with dual group $\hat{G}$. We denote by $M(G)$ the Banach algebra of all bounded regular Borel measures on $G$ under convolution multiplication and total variation norm. In [2], Taylor showed that there is a compact topological semigroup $S$, called the structure semigroup of $M(G)$, and an order preserving isometry-isomorphism $\theta$ of $M(G)$ into $M(S)$ such that:

1. $\theta(M(G))$ is a weak*-dense $L$-subalgebra of $M(S)$;
2. the maximal ideal space of $M(G)$ is identified with $\hat{S}$, the set of all continuous semicharacters on $S$, and the Gel'fand transform of $\mu \in M(G)$ is given by $\hat{\mu}(f) = \int_S f d\theta \mu$ for $f \in \hat{S}$; \{$\hat{\mu}(f); f \in \hat{S}\}$ is called the spectrum of $\mu$.

$\mu$ is called symmetric if $\mu(f) = \overline{\mu}(f)$ for every $f \in \hat{S}$. Let $K$ be the union of all maximal groups $K_p$ of $S$. Then $K = \{x \in S; |f(x)| = 1 \text{ or } 0 \text{ for every } f \in \hat{S}\}$.

Definition. Let $M^+(G)$ be the set of all $\mu \in M(G)$ such that $\theta \mu$ is concentrated on $K$ but $\theta |\mu|(K_p) = 0$ for every $p \in P$.

In [3], Taylor proposes the following problem concerning $M^+(G)$.

Problem. Are there L.C.A. groups $G$ for which $M^+(G) \neq 0$?

The purpose of this paper is to show the existence of a L.C.A. group $G$ such that $M^+(G) \neq 0$. The following is our main theorem.

Theorem. Let $\mathcal{R}$ be the Bohr compactification of the real line $\mathbb{R}$. Then there exists nonzero $\mu \in M^+_K(\mathcal{R})$ so that $\mu$ is a positive symmetric measure and the spectrum of $\mu$ is a countable set.

We put $\Lambda_n = \{(a_0, a_1, \ldots, a_n); a_0 = 0, a_i = 0 \text{ or } 1 (i = 1, 2, \ldots, n)\}$ (for $n = 0, 1, 2, \ldots$) and $\Lambda = \bigcup_{n=0}^{\infty} \Lambda_n$. For $\alpha \in \Lambda$, we put $|\alpha| = n$ if $\alpha \in \Lambda_n$.

Lemma 1. There exists a countable family $\{E_\alpha\}_{\alpha \in \Lambda}$ such that $E_\alpha$ is a...
subset of $R$ ($\alpha \in \Lambda$) satisfying the following conditions:

1. $E_{\alpha} \subseteq E_{\alpha,0}$ and $E_{\alpha} \subseteq E_{\alpha,1}$ for $\alpha \in \Lambda$;
2. $E_{\alpha,0} \setminus E_{\alpha} \neq \emptyset$ and $E_{\alpha,1} \setminus E_{\alpha} \neq \emptyset$ for $\alpha \in \Lambda$;
3. for $\alpha, \beta \in \Lambda$, $E_{\alpha} \cap E_{\beta} = E_{\alpha,0, a_1, \ldots, a_j}$ if $\alpha = (\beta_1, \ldots, \beta_j)$ and $\alpha_{j+1} \neq \beta_{j+1}$;
4. $\bigcup_{\alpha \in \Lambda} E_{\alpha}$ is an independent set.

**Proof.** Since $R$ contains an infinite independent set, it is easy to construct such a family.

We denote by $H_{\alpha}$ the subgroup of $R$ generated by $E_{\alpha}$ ($\alpha \in \Lambda$). The following lemma is clear by Lemma 1.

**Lemma 2.** The countable family $\{H_{\alpha}\}_{\alpha \in \Lambda}$ has the following properties:

1. $H_{\alpha} \subseteq H_{\alpha,0}$ and $H_{\alpha} \subseteq H_{\alpha,1}$ for $\alpha \in \Lambda$;
2. $H_{\alpha,0}/H_{\alpha}$ and $H_{\alpha,1}/H_{\alpha}$ are infinite subgroups for $\alpha \in \Lambda$;
3. for $\alpha, \beta \in \Lambda$, $H_{\alpha} \cap H_{\beta} = H_{\alpha,0, a_1, \ldots, a_j}$ if $\alpha = (\beta_1, \ldots, \beta_j)$ and $\alpha_{j+1} \neq \beta_{j+1}$.

Let $G_{\alpha}$ be the annihilator in $\overline{R}$ of $H_{\alpha}$ ($\alpha \in \Lambda$). We put $G_{\alpha} + G_{\beta} = \{x + y; x \in G_{\alpha}, y \in G_{\beta}\}$; then $G_{\alpha} + G_{\beta}$ is a compact subgroup. The following lemma is clear by Lemma 2.

**Lemma 3.** $\{G_{\alpha}\}_{\alpha \in \Lambda}$ is a family of compact subgroups of $\overline{R}$ and has the following properties:

1. $G_{\alpha} \supseteq G_{\alpha,0}$ and $G_{\alpha} \supseteq G_{\alpha,1}$ for $\alpha \in \Lambda$;
2. $G_{\alpha}/G_{\alpha,0}$ and $G_{\alpha}/G_{\alpha,1}$ are compact infinite groups for $\alpha \in \Lambda$;
3. for $\alpha, \beta \in \Lambda$, $G_{\alpha} + G_{\beta} = G_{\alpha,0, a_1, \ldots, a_j}$ if $\alpha = (\beta_1, \ldots, \beta_j)$ and $\alpha_{j+1} \neq \beta_{j+1}$.

For a compact subgroup $X \subset \overline{R}$, we denote by $m_X$ the normalized Haar measure on $X$. We can consider $m_X \in M(\overline{R})$. We put $\mu_n = (\lambda^n/\sigma \sum_{\alpha \in \Lambda} m_{G_{\alpha}})$, $n = 0, 1, \ldots$. Then $\mu_n \in M(\overline{R})$, $\mu_n \geq 0$ and $\|\mu_n\| = 1$. For $\mu \in M(\overline{R})$, we denote by $\hat{\mu}$ the Fourier-Stieltjes transform of $\mu$. By the definition of $\{\mu_n\}_{n=0}^{\infty}$ and Lemma 2, we get

**Lemma 4.** $\{\mu_n\}_{n=0}^{\infty}$ has the following properties:

1. If $y \in H_0$, then $\mu_n(y) = 1$ for $n = 0, 1, 2, \ldots$;
2. if $y \in H_{a_0, a_1, \ldots, a_{k-1}} \setminus H_{a_0, a_1, \ldots, a_{k-1}}$, then $\mu_n(y) = (\lambda/\sigma)^k$ for $n \geq k$ and $\mu_n(y) = 0$ for $n < k$;
3. if $y \in R \setminus H_{\alpha}$ for every $\alpha \in \Lambda$, then $\mu_n(y) = 0$, $n = 0, 1, 2, \ldots$.

By Lemma 4, $\{\mu_n\}_{n=0}^{\infty}$ has only one weak*-cluster point $\mu$ in $M(\overline{R})$ and has the following properties.

**Lemma 5.** (1) $\mu \in M(\overline{R})$, $\mu \geq 0$ and $\|\mu\| = 1$;
2. if $y \in H_0$, then $\mu_n(y) = 1$.
For \( a \in \Lambda \), we put \( \Lambda_a = \{ \beta \in \Lambda_n; \alpha_0 = \beta, \ldots, \alpha_n = \beta \} \) for \( n \geq |a| \) and \( \Lambda^* = \bigcup_{n \geq |a|} \Lambda_a \). We put \( \mu_n^a = \sum_{\beta \in \Lambda_n} (1/2)^n m_{G_{\beta}} \) for \( n \geq |a| \). Then \( \mu_n^a \geq 0 \), \( \|\mu_n^a\| = (1/2)^n |a| \) and \( \mu_{n+\infty}^a = 1 \) has only one weak*-cluster point \( \mu \in \mathcal{M}(\hat{R}) \), and \( \mu_{n+\infty}^a \) has the following properties.

Lemma 6. (1) \( \mu = \sum_{\beta \in \Lambda_n^*} \mu_n^a \); 
(2) if \( y \in H_{a^*} \), then \( \mu_n(\gamma) = (1/2)^n |a| \); 
(3) for \( y \in H_{a^*}, y \in \Lambda_n^* \) \( (k \geq |a|) \), 
\[ \mu_n^a(y) = (1/2)^k \text{ if } (\beta_0, \beta_1, \ldots, \beta_k) \in \Lambda^a \] 
\[ = 0 \text{ if } (\beta_0, \beta_1, \ldots, \beta_k) \not\in \Lambda^a \] 
(4) if \( y \in R \setminus H_n \) for every \( a \in \Lambda \), then \( \mu_n^a(y) = 0 \).

For a compact subgroup \( X \subset \hat{R} \), there exists the strongest L.C.A. group topology on \( \hat{R} \) such that \( X \) is an open compact subgroup of \( \hat{R} \). We denote by \( R_X \) the resulting L.C.A. topological group. We may consider \( M(R_X) \subset M(\hat{R}) \).

For \( \lambda_1, \lambda_2 \in M(\hat{R}) \), we denote by \( \lambda_1 \perp \lambda_2 \) if \( \lambda_1 \) is mutually singular with \( \lambda_2 \).

For \( \lambda \in M(\hat{R}) \) and a subset \( N \subset M(\hat{R}) \), we denote by \( \lambda \perp N \) if \( \lambda \perp \nu \) for every \( \nu \in N \).

Lemma 7. (1) \( \mu = \sum_{\alpha \in \Lambda} \mu^a \) for every positive integer \( k \); 
(2) \( \mu^a \in M(\hat{R}_{G_\lambda}) \), and \( \mu^a \perp M(\hat{R}_{G_\lambda}) \) for \( \beta \neq \alpha, |\beta| = |\alpha| \); 
(3) for \( \alpha \neq \beta \), 
\[ \mu^a \ast \mu^\beta = (1/2)^{|a|} (1/2)^{|\beta|} m_{G_{\alpha^0, a_1, \ldots, a_j}} \] 
if \( \alpha_1 = \beta_1, \ldots, \alpha_j = \beta_j \) and \( \alpha_{j+1} \neq \beta_{j+1} \).

Proof. By (3) of Lemmas 2, 5 and 6, we have \( \mu = \sum_{\alpha \in \Lambda} \mu^a \) for every integer \( k \), that is \( \mu = \sum_{\alpha \in \Lambda} \mu^a \). For \( \alpha \neq \beta \) such that \( \alpha_1 = \beta_1, \ldots, \alpha_j = \beta_j \) and \( \alpha_{j+1} \neq \beta_{j+1} \), we have 
\[ \mu^a \ast \mu^\beta(y) = \mu^a(y) \mu^\beta(y) = (1/2)^{|a|} \ast (1/2)^{|\beta|} \] 
if \( y \in H_{a_0, \ldots, a_j} \) 
\[ = 0 \] 
if \( y \not\in H_{a_0, \ldots, a_j} \)
by (3) of Lemmas 2 and 6. This shows that 
\[ \mu^a \ast \mu^\beta = (1/2)^{|a|} \ast (1/2)^{|\beta|} m_{G_{\alpha^0, a_1, \ldots, a_j}} \] 

Let \( \phi \) be a canonical homomorphism of \( \hat{R} \) onto \( \hat{R}_{G_\lambda} \), and for \( \lambda \in M(\hat{R}) \) we put \( \hat{\lambda}(E) = \hat{\lambda}(\phi^{-1}(E)) \) for every Borel set \( E \) of \( \hat{R}_{G_\lambda} \). Then \( \hat{\lambda} \in M(\hat{R}_{G_\lambda}) \) and
\( \lambda(y \circ \phi) = \lambda(y) \) for \( y \in \hat{G}/G_a = H^a \). If \( y \in H^a(\alpha = (a_0, a_1, \ldots, a_k)) \) then \( y \in H_{a_0, a_1, \ldots, a_j} \) for some \( 0 \leq j \leq k \) and \( \widehat{\mu}(y) = (\frac{1}{2})^j \) by (3) of Lemma 5. Then we have

\[
\mu = \frac{1}{2} m_{D_{a_0}} + (\frac{1}{2})^2 m_{D_{a_0}, a_1} + \cdots + (\frac{1}{2})^k m_{D_{a_0}, a_1, \ldots, a_{k-1}} + (\frac{1}{2})^k \delta_0,
\]

where \( D_{a_0, a_1, \ldots, a_j} \) is the annihilator in \( \hat{G}/G_a \) of \( H_{a_0, a_1, \ldots, a_j} \subset H^a \),

\( m_{D_{a_0}, a_1, \ldots, a_j} \) is the normalized Haar measure on \( D_{a_0, a_1, \ldots, a_j} \) and \( \delta_0 \) is the point measure at \( 0 \in \hat{G}/G_a \). Since \( H_a/H_{a_0, a_1, \ldots, a_j} \) is an infinite group,

\( m_{D_{a_0}, a_1, \ldots, a_j} \) is a continuous measure on \( \hat{G}/G_a \). Since \( \mu^a_n (n \geq |\alpha|) \) is concentrated on \( G_a \), \( \mu^a \) is concentrated on \( G_a \) and \( \mu^a = (\frac{1}{2})^k \delta_0 \). Thus

\[
\sum_{\beta = a, \beta \in A, |\beta| = |\alpha|} \mu^\beta
\]

is a continuous measure on \( \hat{G}/G_a \) and we have \( \mu^a \in M(\hat{G}/G_a) \) and \( \mu^a \perp M(G_R) \) for \( \beta \neq \alpha \) and \( |\beta| = |\alpha| \).

Remark. By (2) of Lemma 7, \( \mu^a \perp \mu^\beta \) if \( \alpha \neq \beta \) and \( |\alpha| = |\beta| \).

By Lemma 7, we have

**Proposition 1.** \( \theta_{\mu}(K_p) = 0 \) for every maximal group \( K_p \) of \( S \).

**Proof.** Suppose \( \theta_{\mu}(K_p) \neq 0 \) for a maximal group \( K_p \) of \( S \). Then there is a positive integer \( n \) such that \( (\frac{1}{2})^n < \theta_{\mu}(K_p) \). By (1) of Lemma 7, there is an \( \alpha \in \Lambda_n \) such that \( \theta_{\mu^a}(K_p) \neq 0 \). By (2) of Lemma 7, we have \( \theta_{\mu^\beta}(K_p) = 0 \) for every \( \beta \in \Lambda_n (\beta \neq \alpha) \). So we have \( \theta_{\mu}(K_p) = \theta_{\mu^a}(K_p) \leq ||\mu^a|| = (\frac{1}{2})^n \), a contradiction.

For \( f \in \hat{S} \) and \( f^2 = f \), we put \( S_0(f) = \{ x \in S; f(x) = 0 \} \), \( S_1(f) = \{ x \in S; f(x) = 1 \} \) and \( M(S_0(f)) = \{ \mu \in M(G_R); \theta_{\mu} \text{ is concentrated on } S_j(f) \} (j = 0, 1) \). Then \( M(S_0(f)) \) is an \( L \)-ideal of \( M(G_R) \) and \( M(S_1(f)) \) is an \( L \)-subalgebra [2].

**Lemma 8.** Let \( f \in \hat{S} \) such that \( f^2 = f \) and \( \widehat{\mu}(f) \neq 0 \). Then there exists \( \alpha \in \Lambda_n \) such that:

(1) \( \mu^\alpha \in M(S_1(f)) \);

(2) \( \mu^\beta \in M(S_0(f)) \) for \( \beta \neq \alpha \) and \( |\beta| = |\alpha| \).

**Proof.** Since \( \widehat{\mu}(f) \neq 0 \), we can decompose \( \mu = \lambda_1 + \lambda_2 \) (\( \lambda_2 \neq 0 \)), where \( \lambda_1 \in M(S_0(f)) \) and \( \lambda_2 \in M(S_1(f)) \). Suppose that \( \mu_n \in M(S_0(f)) \) for every integer \( n \). For some integer \( n_0 \) such that \( (\frac{1}{2})^{n_0} < ||\lambda_2|| \), there exists \( \chi, \nu \in \Lambda_{n_0} \) such that \( \mu^\chi \perp \lambda_2 \) and \( \mu^\nu \perp \lambda_2 \). Because, \( \mu^\chi \perp \mu^\nu \) for \( \chi, \nu \in \Lambda_{n_0} \) and \( \chi \neq \nu \), by the remark of Lemma 7, and \( ||\mu^\chi|| = (\frac{1}{2})^{n_0} \) for every \( \chi \in \Lambda_{n_0} \).

By Lemma 7, we have

\[
\mu^\chi \perp \mu^\nu = (\frac{1}{2})^{n_0} \mu^\chi_{x_0, x_1, \ldots, x_j},
\]

where \( x_1 = \nu_1, \ldots, x_j = \nu_j \) and \( x_{j+1} \neq \nu_{j+1} \), and \( \mu^\chi \perp \mu^\nu \in M(S_0(f)) \). Since \( \mu^\chi \perp \lambda_2 \) and \( \mu^\nu \perp \lambda_2 \), we have \( \mu^\chi \perp \lambda_2 \perp \lambda_2 = 2^k \lambda_2 \). Since \( \lambda_2 \perp \lambda_2 \in M(S_1(f)) \), we have \( \mu^\chi \perp \mu^\nu \perp \lambda_2 \notin M(S_0(f)) \), a contradiction. Thus there exists an integer \( n \) such that \( \mu_n \notin M(S_0(f)) \). Let \( n_1 \) be the smallest integer such that \( \mu_n \notin M(S_0(f)) \). Then there exists a \( \alpha_1 \) such that \( \mu_{\alpha_1} \in M(S_1(f)) \) and \( m_{G_\beta} \in M(G_R) \).
$M(S_0(f))$ for $\beta \in \Lambda_{n_1}$ and $\beta \neq \alpha$, by (3) of Lemma 3. Since $M(\overline{R}_{G_\alpha}) \subset M(S_1(f))$, we have $\mu^a \in M(S_1(f))$ by (2) of Lemma 7. Suppose that $\mu^\beta \notin M(S_0(f))$ for some $\beta \in \Lambda_{n_1}$ and $\beta \neq \alpha$. Then we have $\mu^\beta \cdot \mu^a \notin M(S_0(f))$.

By (3) of Lemma 7, we have

$$\mu^\beta \cdot \mu^a = (\beta_j | \beta_i) + \frac{1}{m_{G_{a_0}, a_1}, \ldots, a_j},$$

where $\alpha_1 = \beta_1, \ldots, \alpha_j = \beta_j$, and $\alpha_{j+1} \neq \beta_{j+1}$, and $j < |\alpha| = n_1$. This shows that $\mu^\beta \cdot \mu^a \notin M(S_0(f))$, a contradiction. Thus we complete this lemma.

For $f \in \hat{S}$, $f \geq 0$, there exists $g_f \in \hat{S}$, $g_f^2 = g_f$ such that $M(S_1(g_f)) = M(O(f))$, where $O(f) = \{x \in S; f(x) = 1\}$ and $M(O(f)) = \{\mu \in M(\overline{R}); \theta \mu \text{ is concentrated on } O(f)\}$. [2]

**Proposition 2.** $\theta \mu$ is concentrated on $K$.

**Proof.** Let $f \in \hat{S}$ such that $f \geq 0$, $f \neq 1$ and $\hat{\eta}(f) \neq 0$. Let $f = h_f \cdot f$ be the polar decomposition of $f$, where $h_f \cdot h_f = 1$ [2, Lemma 3.3]. Then $\hat{\mu}(h_f) \neq 0$. By Lemma 8, there exists $\alpha \in \Lambda$ such that $\mu^\alpha \in M(S_1(h_f))$ and $\mu^\beta \in M(S_0(h_f))$ for $\beta \neq \alpha$ and $|\beta| = |\alpha|$. Since $M(\overline{R}_{G_{a_\alpha}}) \subset M(S_1(h_f))$ and $m_{G_{a_\alpha}} \in M(S_1(g_f))$, we have $M(\overline{R}_{G_{a_\alpha}}) \subset M(g_f))$ and $\mu^\alpha \in M(S_1(g_f))$. Thus we complete the proof of this proposition.

**Proposition 3.** $\mu$ is a symmetric measure and $\mu$ has a countable spectrum.

**Proof.** Since $\hat{\mu} \geq 0$, we have $\mu^* = \mu$. Let $f \in \hat{S}$. By the proof of Proposition 2, there exists $\alpha \in \Lambda$ such that $\hat{\mu}(f) = \mu^\alpha(f)$. Since $\mu^\alpha \in M(\overline{R}_{G_{a_\alpha}})$ and $\hat{\lambda}(f) = ||\lambda||$ for every positive $\lambda \in M(\overline{R}_{G_{a_\alpha}})$, there exists $\gamma \in \overline{G_{a_\alpha}}$ such that $\hat{\mu}(f) = \mu^\alpha(f) = \hat{\mu}(\gamma) = \hat{\mu}(\gamma)$. By Lemma 6, we have

$$\{\mu^\gamma(\eta); \eta \in R_{\gamma} = 1, (\frac{1}{2^0})|\alpha_1|, (\frac{1}{2^1})|\alpha_1|+1, (\frac{1}{2^2})|\alpha_1|+2, \ldots\}.$$

Thus we have $\{\mu^\gamma(\eta); f \in \hat{S} = \{0, 1, \frac{1}{2}, (\frac{1}{2})^2, (\frac{1}{2})^3, \ldots\}$. This shows that $\mu$ is symmetric and has a countable spectrum.

By Propositions 1, 2 and 3, we have our Theorem.

**Corollary.** There is a compact metrizable abelian group $G$ and a nonzero symmetric measure $\mu \in M_K(G)$ so that the spectrum of $\mu$ is a countable set.

**Proof.** We may assume that $E_\alpha$ is a countable set ($\alpha \in \Lambda$) in Lemma 1. Let $H$ be the subgroup generated by $\{H_{a_\alpha}; \alpha \in \Lambda\}$, then $H$ is a countable subgroup of $R$. Let $H^\perp$ be the annihilator of $H$ in $\overline{R}$; then $H^\perp$ is a compact subgroup of $\overline{R}$. Since $H = \overline{R}/H^\perp$, $\overline{R}/H^\perp$ is a compact metrizable group. Then we can construct $\mu \in M(\overline{R}/H^\perp)$, which has the properties of this corollary, in the same way as in the proof of our Theorem.
REFERENCES


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