

## CARTAN SUBALGEBRAS AND TORI IN PRIME CHARACTERISTIC

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ABSTRACT. Let  $L$  be a finite dimensional (not necessarily restricted) Lie algebra over a perfect field  $F$  of characteristic  $p > 0$ . Let  $\bar{L}$  be any finite dimensional restricted Lie algebra containing  $L$ . The Cartan subalgebras of  $L$  are characterized as the centralizers of certain tori in  $\bar{L}$ .

Let  $\bar{L}$  be a finite dimensional restricted Lie algebra over a perfect field  $F$  of characteristic  $p > 0$ . Seligman [5, Chapter V.7] (and, independently, Schue [4]) has given definitions of semisimple and nilpotent elements of  $\bar{L}$  ( $x \in \bar{L}$  is semisimple if  $x$  is a linear combination of  $\{x^{p^i} \mid i \geq 1\}$ , and is nilpotent if  $x^{p^i} = 0$  for some  $i$ ) and has shown the existence of a Jordan-Chevalley decomposition of an element  $x \in \bar{L}$  into its semisimple and nilpotent parts. (Thus  $x = x_s + x_n$  where  $x_s$  is semisimple and  $x_n$  is nilpotent. Furthermore,  $x_s$  and  $x_n$  are both  $p$ -polynomials in  $x$ , and the sum of two commuting semisimple (respectively nilpotent) elements is semisimple (respectively nilpotent).) Defining a torus  $T \subseteq \bar{L}$  to be an abelian subalgebra consisting entirely of semisimple elements, Seligman proves that  $H$  is a Cartan subalgebra of  $\bar{L}$  if and only if  $H$  is the centralizer of a maximal torus in  $\bar{L}$ .

This characterization of Cartan subalgebras has proven to be extremely useful. For example, it reduces questions of conjugacy for Cartan subalgebras to questions of conjugacy for tori. This makes possible Demuškin's determination [1], [2] of the conjugacy classes of Cartan subalgebras of the simple restricted Lie algebras of Cartan type.

In this paper we will prove an extension of Seligman's theorem to arbitrary finite dimensional Lie algebras (not necessarily restricted) over  $F$ . Our result depends on definitions of certain types of tori of  $\bar{L}$ . These definitions are given in §1. Our main result, which characterizes the Cartan subalgebras, is stated and proved in §2.

**I. Compatible and saturated tori.** Let  $F$  be a perfect field of characteristic  $p > 0$ . We write  $N_A(B)$  for the normalizer of  $B$  in  $A$  and  $C_A(B)$  for the centralizer of  $B$  in  $A$ .

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**Lemma.** *Let  $L$  be a finite dimensional Lie algebra over  $F$ . Then there is a finite dimensional restricted Lie algebra  $\bar{L}$  with  $L \subseteq \bar{L}$ .*

**Proof.** By Iwasawa's theorem [3, p. 204]  $L$  has a faithful representation on a finite dimensional space  $V$ . Then we may take  $\bar{L} = \text{End } V$  (or any restricted subalgebra of  $\text{End } V$  which contains  $L$ ).

**Definitions.** Let  $\bar{L}$  be a restricted Lie algebra,  $L$  a (not necessarily restricted) subalgebra. We say a torus  $T \subseteq \bar{L}$  is *compatible* with respect to  $L$  if  $T$  is spanned by elements of the form  $x_s$  where  $x \in C_L(T)$ . We say  $T$  is *saturated* with respect to  $L$  if  $x \in C_L(T)$  implies  $x_s \in C_{\bar{L}}(C_L(T))$ .

Note that if  $L = \bar{L}$ , then a maximal torus in  $L$  is both compatible and saturated with respect to  $L$ .

## 2. Characterization of Cartan subalgebras.

**Theorem.** *Let  $L$  be a finite dimensional Lie algebra over a perfect field  $F$  of characteristic  $p > 0$ . Let  $\bar{L}$  be a finite dimensional restricted Lie algebra containing  $L$ .*

(a) *Let  $T$  be a torus of  $\bar{L}$  which is compatible and saturated with respect to  $L$ . Then  $C_L(T)$  is a Cartan subalgebra of  $L$ .*

(b) *Let  $H$  be a Cartan subalgebra of  $L$  and  $T$  the space spanned by  $\{x_s \mid x \in H\}$ . Then  $T$  is a torus in  $\bar{L}$  which is compatible and saturated with respect to  $L$ . Furthermore,  $H = C_L(T)$ .*

**Proof.** (a) Let  $T$  be a torus of  $\bar{L}$  which is compatible and saturated with respect to  $L$ . Let  $x \in C_L(T)$ . Since  $T$  is saturated with respect to  $L$  we have  $x_s \in C_{\bar{L}}(C_L(T))$ . Thus  $\text{ad } x|_{C_L(T)} = \text{ad } x_n|_{C_L(T)}$  is nilpotent. Thus by Engel's theorem,  $C_L(T)$  is nilpotent. Let  $y \in N_L(C_L(T))$ . If  $x \in C_L(T)$ , then  $y(\text{ad } x)^t = 0$  for sufficiently large  $t$  and, hence,  $y(\text{ad } x_s) = 0$ . Since  $T$  is compatible with respect to  $L$ , hence, spanned by such  $x_s$ , we have  $y \in C_L(T)$ . Thus  $C_L(T)$  is a Cartan subalgebra of  $L$ .

(b) Let  $H$  be a Cartan subalgebra of  $L$ . Since  $H$  is nilpotent, we have  $\text{ad } x_s|_H = 0$  for all  $x \in H$ , hence,  $[x_s, y_s] = 0$  for all  $x, y \in H$ . Thus  $T$  is a torus in  $\bar{L}$ , is compatible with respect to  $L$ , and  $H \subseteq C_L(T)$ . If  $x \in H$ , then  $\text{ad } x|_{C_L(T)} = \text{ad } x_n|_{C_L(T)}$  is nilpotent. Hence, if  $H \neq C_L(T)$ , Engel's theorem, applied to  $C_L(T)/H$ , shows  $N_L(H) \neq H$ , a contradiction. Thus  $H = C_L(T)$ . Finally  $x \in C_L(T) = H$  implies  $x_s \in T \subseteq C_{\bar{L}}(C_L(T))$  so  $T$  is saturated with respect to  $L$ .

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