$p$-EXTREMAL LENGTH AND $p$-MEASURABLE
CURVE FAMILIES

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ABSTRACT. It is well known that the reciprocal of $p$-extremal length,
considered as a set function, is an outer measure. We show that if a curve
family in euclidean $n$-space is measurable with respect to this outer mea-
sure, then the $p$-extremal length of the curve family is zero or infinite.

1. Introduction. Let $\mathcal{C}$ be the totality of curves in $n$-dimensional eucli-
dean space $\mathbb{R}^n$. More precisely, if $\gamma \in \mathcal{C}$ then $\gamma$ is a continuous mapping
of an open interval $(a, b)$ into $\mathbb{R}^n$ where $a, b \in [-\infty, \infty]$. If $p \in (1, \infty)$ and
$\Gamma \subset \mathcal{C}$, the $p$-extremal length of $\Gamma$, denoted by $\Lambda_p(\Gamma)$, is defined as follows.
Let $\mathcal{J}(\Gamma)$ be the set of nonnegative Borel functions $\rho: \mathbb{R}^n \to [0, \infty]$ satisfying
the condition that for every locally rectifiable curve $\gamma \in \Gamma$, the line
integral of $\rho$ with respect to arc length $\int_{\gamma} \rho \, ds \geq 1$. Then
\[
\frac{1}{\Lambda_p(\Gamma)} = \inf_{\rho \in \mathcal{J}(\Gamma)} \int_{\mathbb{R}^n} \rho^p \, dm
\]
where $m_n$ is Lebesgue $n$-measure on $\mathbb{R}^n$ and the infimum is taken over all
$\rho \in \mathcal{J}(\Gamma)$. If $\mathcal{J}(\Gamma) = \emptyset$ we set $\Lambda_p(\Gamma) = 0$. The $p$-modulus of $\Gamma$, denoted by
$M_p(\Gamma)$, is defined as the reciprocal of the $p$-extremal length of $\Gamma$.

It is well known [1, Theorems 1 or 3, Theorem 6.2] that $M_p$, regarded
as a set function on $\mathcal{C}$, is an outer measure. That is, 1. $M_p(\emptyset) = 0$, 2.
$\Gamma_1 \subset \Gamma_2 \subset \mathcal{C}$ implies $M_p(\Gamma_1) \leq M_p(\Gamma_2)$, and 3. if $\Gamma_i \subset \mathcal{C}$, $i = 1, 2, \ldots$, then
$M_p(\bigcup_{i=1}^{\infty} \Gamma_i) \leq \sum_{i=1}^{\infty} M_p(\Gamma_i)$. It appears reasonable to ask the question:
What subsets of $\mathcal{C}$ are $p$-measurable? In other words, for which curve fami-
lies $\Gamma \subset \mathcal{C}$ do we have
\[
M_p(\theta) = M_p(\theta \cap \Gamma) + M_p(\theta - \Gamma)
\]
for all $\theta \in \mathcal{C}$? Renggli [2] has shown that if $p = 2$, $n = 2$, and $0 < M_2(\Gamma)$
$< \infty$, then $\Gamma$ is not 2-measurable. It is the purpose of this paper to simplify
and generalize this result for all $p \in (1, \infty)$ and all $n \geq 2$. In particular, no
use is made of conformal mappings.

2. Preliminary lemmas.

2.1. Lemma. Let $E \subset \mathbb{R}^{n-1}$ and $h \in (0, \infty)$. Let $\Gamma$ be the family of
curves $\gamma: (0, h) \to \mathbb{R}^n$ such that for $t \in (0, h)$, $\gamma(t) = (x_1, \ldots, x_{n-1}, t)$
where \((x_1, \ldots, x_{n-1}) \in E\). Then \(M_p(\Gamma) = m_n^*(E)/h^{p-1}\) where \(m_n^*(E)\) is the \((n-1)\)-dimensional Lebesgue outer measure of \(E\). Furthermore, if \(E\) is an \((n-1)\)-Borel measurable set then the function \(\rho_0: \mathbb{R}^n \to [0, \infty]\) defined by

\[
\rho_0(x_1, \ldots, x_n) = \begin{cases} 
1/h & \text{if } (x_1, \ldots, x_{n-1}) \in E \text{ and } 0 < x_n < h, \\
0 & \text{otherwise}, 
\end{cases}
\]

is in \(\mathcal{F}(\Gamma)\) and \(M_p(\Gamma) = \int_{\mathbb{R}^n} \rho_0^p \, dm\).

**Proof.** Let \(V \subseteq \mathbb{R}^{n-1}\) be an open set containing \(E\). Let \(p: \mathbb{R}^n \to [0, \infty]\) be the Borel function defined by

\[
\rho(x_1, \ldots, x_n) = \begin{cases} 
1/h & \text{if } (x_1, \ldots, x_{n-1}) \in V \text{ and } 0 < x_n < h, \\
0 & \text{otherwise}. 
\end{cases}
\]

Clearly \(\rho \in \mathcal{F}(\Gamma)\). Therefore \(M_p(\Gamma) \leq \int_{\mathbb{R}^n} p^p \, dm \leq m_n^*(V)/h^{p-1}\). Since this is valid for every open set \(V \subseteq \mathbb{R}^{n-1}\) which contains \(E\), we get \(M_p(\Gamma) \leq m_n^*(E)/h^{p-1}\). We now proceed to establish the reverse inequality. Let \(\rho \in \mathcal{F}(\Gamma)\). Then \(1 \leq \int_0^h \rho(x_1, \ldots, x_{n-1}, t) \, dt\) for all \((x_1, \ldots, x_{n-1}) \in E\). An application of Hölder's inequality yields

\[
h^{1-p} \leq \int_0^h \rho^p(x_1, \ldots, x_{n-1}, t) \, dt
\]

for all \((x_1, \ldots, x_{n-1}) \in E\). Let \(F\) be the set of \((x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}\) where \(h^{1-p} \leq \int_0^h \rho^p(x_1, \ldots, x_{n-1}, t) \, dt\). Then \(F\) is \((n-1)\)-measurable and \(E \subseteq F\). Therefore

\[
\int_F \int_0^h \rho^p(x_1, \ldots, x_{n-1}, t) \, dt \, dm_{n-1} \leq \int_{\mathbb{R}^n} \rho^p \, dm.
\]

Since \(\rho \in \mathcal{F}(\Gamma)\) is arbitrary, we get \(m_n^*(E)/h^{p-1} \leq M_p(\Gamma)\).

The other part of the lemma is clear.

2.2. **Comment.** In a way which can easily be made precise, the conclusions of the lemma remain correct if \(E\), \(\Gamma\) and \(\rho_0\) are "translated" by an amount \(b \in \mathbb{R}^n\).

The following lemma will be needed. The proof follows from [1, (e), p. 178] or [3, Theorem 6.7].

2.3. **Lemma.** Let \(E_1, \ldots, E_k\) be disjoint Borel sets in \(\mathbb{R}^n\). Let \(\Gamma_1, \ldots, \Gamma_k\) be curve families such that every curve in \(\Gamma_i\) lies in \(E_i\), \(i = 1, \ldots, k\). Then \(M_p(\bigcup_{i=1}^k \Gamma_i) = \sum_{i=1}^k M_p(\Gamma_i)\).

3. **Main lemma and theorem.**

3.1. **Lemma.** For every pair of positive integers \(j, k\) there exists a curve family \(\Gamma_{j,k} \subseteq \mathcal{C}\) with the following properties: (i) \(M_p(\Gamma_{j,k}) < \infty\), (ii) every curve in \(\Gamma_{j,k}\) lies in the closed cube \(S_k = \{(x_1, \ldots, x_n) \in \mathbb{R}^n: |x_i| < 1\}\) (iii)
where \( \rho_{j,k} \in \mathcal{F}(\Gamma_{i,k}) \) is defined by

\[
\rho_{j,k}(x) = \begin{cases} 
1 & \text{if } x \in S_k, \\
0 & \text{if } x \in R^n - S_k,
\end{cases}
\]

and (iv) if \( \Gamma \subset \mathcal{C} \) then

\[
M_p(\Gamma_{i,k} \cap \Gamma) + (1 - 2^{1-p})M_p(\Gamma_{2j,k} \cap \Gamma) \leq M_p(\Gamma).
\]

**Proof.** Let \( P_{i,k}^j \) be the bounded hyperplane defined by

\[
P_{i,k}^j = \{(x_1, \ldots, x_n) \in R^n; x_n = j \} \cap S_k, \quad i = -k_j, -(k_j + 1), \ldots, k_j.
\]

Let \( \Gamma_{i,k}^j \) be the family of curves \( y: (0, 1/j) \to S_k \) such that for \( t \in (0, 1/j) \), \( y(t) = (x_1, \ldots, x_{n-1}, i/j + t) \) where \(|x_j| \leq k, l = 1, \ldots, n-1. \) \( \Gamma_{j,k}^i \) is the family of segments in \( S_k \) which "connect" the planes \( P_{j,k}^i \) and \( P_{j,k}^{i+1} \) and which are parallel to the \( x_n \)-axis. Let \( \Gamma_{j,k}^n = \bigcup_{i=-k_j}^{k_j} \Gamma_{j,k}^i \). Parts (i), (ii), and (iii) of the lemma follow from 2.1, 2.2, and 2.3.

We proceed to prove part (iv). Define \( \Pi_j: S_k \to \bigcup_{i=-k_j}^{k_j} P_{i,k}^j \) as follows. If \((x_1, \ldots, x_n) \in S_k\) then for some \( i \in \{ -k_j, \ldots, k_j \}\) we have \( i/j \leq x_n < (i + 1)/j \). Set \( \Pi_j(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}, i/j). \) Let \( U \) be the set of points \( \Pi_{2j,k}(x) \) where \( x \) is a point on a curve in \( \Gamma_{2j,k} \cap \Gamma \). Lemma 2.1 gives

\[
M_p(\Gamma_{2j,k} \cap \Gamma) = m_{n-1}^*(U)(2j)^{p-1}.
\]

Let \( \epsilon \in (0, 1) \) and choose a relatively open set \( V \) in \( \bigcup_{i=-2k_j}^{2k_j} P_{i,k}^j \) such that \( U \subset V \) and

\[
m_{n-1}^*(V) \leq m_{n-1}^*(U) + \epsilon.
\]

Write \( \Gamma_{j,k} \cap \Gamma = \alpha \cup \beta \) where \( \beta \) is the set of curves in \( \Gamma_{j,k} \cap \Gamma \) which contain a point \( x \) such that \( \Pi_{2j,k}(x) \in V \) and \( \alpha \) is the complement of \( \beta \) in \( \Gamma_{j,k} \cap \Gamma \). Lemma 2.1 gives

\[
M_p(\beta) \leq m_{n-1}^*(V)^{p-1}.
\]

Since the curves in \( \alpha \) and \( \Gamma_{2j,k} \cap \Gamma \) lie in disjoint Borel sets, we conclude

\[
M_p(\alpha) + M_p(\Gamma_{2j,k} \cap \Gamma) = M_p(\alpha \cup (\Gamma_{2j,k} \cap \Gamma)) \leq M_p(\Gamma).
\]

Therefore,

\[
M_p(\alpha) \leq M_p(\Gamma) - M_p(\Gamma_{2j,k} \cap \Gamma).
\]
\[ M_p(\Gamma_j, k \cap \Gamma) = M_p(\alpha \cup \beta) \leq M_p(\alpha) + M_p(\beta). \]

The above and relations 3.5–3.8 give
\[ M_p(\Gamma_j, k \cap \Gamma) \leq M_p(\Gamma_j, k \cap \Gamma) - M_p(\Gamma_j, k \cap \Gamma) + \frac{M_p(\Gamma_j, k \cap \Gamma)}{(2^2)^{p-1} + \epsilon}]^{p-1}. \]

Letting \( \epsilon \to 0 \) in the above and simplifying, we get 3.4.

**3.9. Theorem.** Let \( \Gamma \subseteq \mathbb{C} \) and assume \( 0 < M_p(\Gamma) < \infty \). Then \( \Gamma \) is not \( p \)-measurable.

**Proof.** We assume that \( \Gamma \) is \( p \)-measurable and proceed to obtain a contradiction.

Choose \( \rho \in \mathcal{F}(\Gamma) \) such that \( \int_{\mathbb{R}^n} \rho_n^p \, dm_n < \infty \) and \( \epsilon \in (0, 1) \). Let \( k \) be an integer such that \( \int_{\mathbb{R}^n-S_k} \rho_n^p \, dm_n < \epsilon/2 \). Let \( j \) be an integer such that \( \mathcal{J}_\mathcal{L}(\Sigma_k, \rho_n^p \, dm_n < \epsilon/2 \). Let \( \rho' = \max(\rho, \rho_j, k) \). Then \( \rho' \in \mathcal{F}(\Gamma \cup \Gamma_j, k) \).

Therefore,
\[
M_p(\Gamma \cup \Gamma_j, k) - M_p(\Gamma_j, k) \leq \int_{\mathbb{R}^n} \rho_n^p \, dm_n - \int_{\mathbb{R}^n} \rho_n^p \, dm_n
\]
\[
\leq \int_{\mathbb{R}^n} \rho_n^p \, dm_n - \int_{\mathbb{R}^n} \rho_n^p \, dm_n
\]
\[
\leq \epsilon/2 + \epsilon/2 = \epsilon.
\]

Therefore,
\[
M_p(\Gamma \cup \Gamma_j, k) \leq M_p(\Gamma_j, k) + \epsilon.
\]

In exactly the same way we get
\[
M_p(\Gamma \cup \Gamma_j, k) \leq M_p(\Gamma_j, k) + \epsilon.
\]

We now use the assumption of the \( p \)-measurability of \( \Gamma \) to get
\[
M_p(\Gamma_j, k) = M_p(\Gamma_j, k \cap \Gamma) + M_p(\Gamma_j, k - \Gamma)
\]
and
\[
M_p(\Gamma_j, k \cup \Gamma) = M_p(\Gamma) + M_p(\Gamma_j, k - \Gamma).
\]

The above equations imply
\[
M_p(\Gamma_j, k) + M_p(\Gamma) = M_p(\Gamma_j, k \cup \Gamma) + M_p(\Gamma_j, k \cap \Gamma).
\]

The above and 3.10 give
\[
M_p(\Gamma) \leq M_p(\Gamma_j, k \cap \Gamma) + \epsilon.
\]

In exactly the same way we get
Relations 3.4, 3.12, and 3.13 give
\[(M_p(\Gamma) - \epsilon) + (1 - 2^{1-p})(M_p(\Gamma) - \epsilon) \leq M_p(\Gamma).\]
Letting \(\epsilon \to 0\) in the above allows us to conclude \(M_p(\Gamma) \leq 0\). This contradicts the hypothesis that \(0 < M_p(\Gamma)\).

4. Conclusion.

**Theorem.** Let \(\Gamma \subset \overline{C}\). Then (a) if \(M_p(\Gamma) = 0\), then \(\Gamma\) is \(p\)-measurable; (b) if \(0 < M_p(\Gamma) < \infty\), then \(\Gamma\) is not \(p\)-measurable; and (c) if \(M_p(\Gamma) = \infty\), then \(\Gamma\) may or may not be \(p\)-measurable.

**Proof.** (a) is obvious and (b) is Theorem 3.9.

If \(M_p(\Gamma) = \infty\) and \(0 < M_p(\overline{C} - \Gamma) < \infty\), then \(\Gamma\) is not \(p\)-measurable since the complement of a measurable set is measurable. If \(M_p(\Gamma) = \infty\) and \(M_p(\overline{C} - \Gamma) = 0\), then \(\Gamma\) is \(p\)-measurable since \(\overline{C} - \Gamma\) is \(p\)-measurable and its complement is \(\Gamma\). It is easy to construct examples to show that both of these possibilities can occur.

REFERENCES