ON THE CONVOLUTION OF A MEASURE AND A FUNCTION

S. K. BERBERIAN

ABSTRACT. Complements to a theorem of Bourbaki on the convolution of a measure and a function.

The setting is as follows [3, Chapter VIII, §4, No. 1]: \( X \) is a locally compact space, \( G \) is a locally compact group acting continuously on the left in \( X \), and \( \beta \) is a nonzero positive measure on \( X \) that is quasi-invariant under \( G \); more precisely,

\[
\gamma(s)\beta = \chi(s^{-1}, \cdot) \cdot \beta
\]

for all \( s \in G \), where \( \chi \) is universally measurable and everywhere \( > 0 \) on \( G \times X \).

The following proposition is central to the discussion of convolution of functions in [3]:

**Proposition** [3, Chapter VIII, §4, No. 1, Proposition 2]. Let \( \mu \) be a measure on \( G \), \( f \) a locally \( \beta \)-integrable complex function on \( X \). Assume that one of the following conditions is verified:

(i) \( f \) and \( \chi \) are continuous;

(ii) \( G \) operates properly in \( X \), and \( f \) is zero on the complement of a denumerable union of compact sets;

(iii) \( \mu \) is carried by a denumerable union of compact sets.

If \( \mu \) and \( f \) are convolvable relative to \( \beta \), then the function \( s \mapsto f(s^{-1}x) \chi(s^{-1}, x) \) is essentially \( \mu \)-integrable for locally \( \beta \)-almost all \( x \); and if \( \mu \ast f \) denotes any locally \( \beta \)-integrable function such that \( (\mu \ast f) \cdot \beta = \mu \ast (f \cdot \beta) \), then

\[
(\mu \ast f)(x) = \int f(s^{-1}x)\chi(s^{-1}, x)\,d\mu(s)
\]

locally \( \beta \)-almost everywhere.

Case (ii) apparently needs an additional hypothesis, and the proof of Case (iii) given in [3] has some sizable gaps. The aim of this paper is to clarify these points; also, we reformulate condition (iii) so as to make it more flexible in applications, one of which is given. All notations and terminology are taken from [1]–[3].

By way of motivation, we remark that the roughly comparable result in
the treatise of E. Hewitt and K. A. Ross is Lemma (20.6) of [5]. There, the function $f$ is assumed to be a Borel function, therefore the function $(s, x) \mapsto f(s^{-1}x)$ is a Borel function on the product space. This smooths the way for an application of Fubini's theorem [5, Theorem (13.9)]; moreover, all of the partial functions $s \mapsto f(s^{-1}x), x \mapsto f(s^{-1}x)$ are Borel functions. Consequently, the measurability problem considered in Lemma 3 below does not arise in [5]; on the other hand, the Bourbaki formulation is more flexible in that non-Borel functions are allowed.

**Lemma 1.** The following conditions on a measure $\mu$ are equivalent:
(a) $\mu$ is carried by a denumerable union of compact sets;
(b) $\mu$ is carried by a denumerable union of essentially $\mu$-integrable sets;
(c) $\mu$ is carried by a $\mu$-moderated, $\mu$-measurable set;
(d) $|\mu| = \sum_1^\infty \mu_n$, where $(\mu_n)$ is a summable sequence of bounded positive measures.

**Proof.** We can suppose $\mu \geq 0$. It is trivial that (a) implies (b).

(b) $\Rightarrow$ (d). Suppose $\mu$ is carried by $S = \bigcup_1^\infty S_n$, where the $S_n$ are essentially $\mu$-integrable and, as we may suppose, disjoint. The measures $\mu_n = \phi_{S_n} \cdot \mu$ ($\phi$ denotes characteristic function) are bounded [2, §5, No. 3, Corollary of Theorem 1]; since $\mu = \phi_S \cdot \mu$ and $\phi_S = \sum_1^\infty \phi_{S_n}$, it is an elementary consequence of the Lebesgue dominated convergence theorem that $\mu = \sum_1^\infty \mu_n$ [1, Chapter IV, §4, No. 3, Corollary 2 of Theorem 2], [2, §2, No. 1].

(d) $\Rightarrow$ (c). Suppose $\mu = \sum_1^\infty \mu_n$, where the $\mu_n$ are bounded positive measures. For each $n$, write $\mu_n = f_n \cdot \mu$ with $f_n$ essentially $\mu$-integrable [2, §5, No. 5, Theorem 2 and No. 3, Corollary of Theorem 1]. We can suppose that $f_n$ is $\mu$-integrable [2, §5, No. 3, Corollary 2 of Proposition 3]. Then for every $n$, the set $A_n = \{s; f_n(s) \neq 0\}$ is $\mu$-measurable, and $\mu$-moderated [2, §1, No. 3, Corollary of Proposition 9], hence so is $A = \bigcup_1^\infty A_n$. For all $n$, evidently $\phi_A \cdot \mu_n = \mu_n$, that is, $CA$ is locally $\mu_n$-negligible; it follows that $CA$ is locally $\mu$-negligible [2, §2, No. 2, Corollary 2 of Proposition 1], in other words $A$ carries $\mu$.

(c) $\Rightarrow$ (a). Suppose $A$ is a $\mu$-moderated, $\mu$-measurable set such that $\phi_A \cdot \mu = \mu$. Then $A \subseteq \bigcup_1^\infty K_n \cup N$, where the $K_n$ are compact and $N$ is $\mu$-negligible. Writing $S = \bigcup_1^\infty K_n$, the relation $CS \subseteq N \cup CA$ shows that $CS$ is locally $\mu$-negligible.

**Lemma 2.** If $f$ is a locally $\beta$-integrable function on $X$ and if $\mu$ is a measure on $G$ such that $\mu$ and $f$ are convolvable relative to $\beta$, then the function $F(s, x) = f(s^{-1}x)\chi(s^{-1}, x)$ on $G \times X$ is measurable for $\mu \otimes \beta$.

**Proof.** For every $h \in K(X)$, the function $(1 \otimes h)F$ is essentially integrable for $\mu \otimes \beta$ [3, Chapter VIII, §4, No. 1, proof of Proposition 2], hence
\( \mu \otimes \beta \)-measurable. Then \( F \) is \( \mu \otimes \beta \)-measurable by a routine application of the principle of localization [1, Chapter IV, §5, No. 2, Proposition 4]. Incidentally, in view of the hypotheses on \( \chi \), it is the same to say that the function \( (s, x) \mapsto f(s^{-1}x) \) is \( \mu \otimes \beta \)-measurable.

**Lemma 3.** With hypotheses as in Lemma 2, let \( M = \{ x : F(., x) \text{ is not } \mu \text{-measurable} \} \).

1. If \( f \) and \( \chi \) are continuous, then \( M = \emptyset \).
2. If \( F \) (equivalently, the function \( (s, x) \mapsto f(s^{-1}x) \)) is moderated for \( \mu \otimes \beta \), then \( M \) is \( \beta \)-negligible.
3. If \( \mu \) is carried by a denumerable union of compact sets, then \( M \) is locally \( \beta \)-negligible.

**Proof.** (1) For every \( x \), \( F(., x) \) is continuous.

(2) In view of Lemma 2, this is immediate from [2, §8, No. 2, Proposition (2a)].

(3) In view of criterion (d) of Lemma 1, this follows from the proof of [2, §8, No. 2, Proposition (2b)].

**Proof of the Proposition** (under an added hypothesis in Case (ii)). We can suppose \( f \geq 0 \) and \( \mu \geq 0 \). Let \( F(s, x) = f(s^{-1}x)\chi(s^{-1}, x) \), and let \( g : X \rightarrow \mathbb{R}^+ \) be the function defined by the formula \( g(x) = \int F(., x) \, d\mu \). One has \( g(x) = \int F(., x) \, d\mu \) in Case (i) (because every \( F(., x) \) is continuous [2, §1, No. 1, Proposition 4]) and in Case (ii) (because, for each \( x \), \( F(., x) \) vanishes outside a denumerable union of compact sets [3, Chapter III, §4, No. 5, Theorem (1b)]).

**Cases (i), (ii).** As shown in [3], \( g \) is locally \( \beta \)-integrable and \( g \cdot \beta = \mu \ast (f \cdot \beta) \), that is, \( g \) is a determination of \( \mu \ast f \). In particular, the set \( N = \{ x : g(x) = +\infty \} \) is locally \( \beta \)-negligible. In Case (i), this means (in view of part (1) of Lemma 3) that \( F(., x) \) is \( \mu \)-integrable for locally \( \beta \)-almost all \( x \).

In Case (ii), if one assumes that the function \( (s, x) \mapsto f(s^{-1}x) \) is \( \mu \otimes \beta \)-moderated, then it results from part (2) of Lemma 3 that \( F(., x) \) is \( \mu \)-integrable for locally \( \beta \)-almost all \( x \).

**Case (iii).** Suppose \( \mu \) is carried by a denumerable union \( S \) of compact sets. Since \( \phi_S = 1 \) locally \( \mu \)-almost everywhere and \( S \) is \( \mu \)-moderated, one has

\[
g(x) = \int F(., x) \, d\mu = \int F(., x) \, d\mu
\]

for all \( x \in X \). As shown in [3], \( g \) is again locally \( \beta \)-integrable and is a determination of \( \mu \ast f \). In particular, \( g(x) < +\infty \) locally \( \beta \)-a.e.; in view of part (3) of Lemma 3, this means that \( F(., x) \) is essentially \( \mu \)-integrable for locally \( \beta \)-almost all \( x \) [2, §1, No. 3, Proposition 9].
The following application is a slight extension of [3, Chapter VIII, §4, No. 5, Proposition 10]:

**Corollary.** Let \( \beta \) be a relatively invariant, nonzero positive measure on \( G \), and let \( f, g \) be locally \( \beta \)-integrable functions on \( G \) such that \( f \) and \( g \) are convolvable relative to \( \beta \). If one of \( f, g \) is continuous or is zero outside a denumerable union of essentially \( \beta \)-integrable sets, then

\[
(f \ast g)(x) = \int g(s^{-1}x) \chi(s) \chi'(s^{-1}) d\beta(s) = \int f(xs^{-1})g(s) \chi'(s^{-1}) d\beta(s)
\]

for locally \( \beta \)-almost all \( x \).

**Proof.** Here \( \chi \) and \( \chi' \) denote the left and right multiplicators of \( \beta \), which are continuous [3, Chapter VII, §1, No. 8]. It is straightforward to show that the two essential integrals (or integrals) exist simultaneously and are then equal, thus it is immaterial whether the conditions are imposed on \( f \) or on \( g \). We can suppose \( f \geq 0, g \geq 0 \). Let \( \mu = f \cdot \beta \).

If \( g \) is continuous, one applies Case (i) of the Proposition to \( p \), \( g \).

Suppose \( f \) is zero outside \( S = \bigcup_{1}^{\infty} A_{n} \), where the \( A_{n} \) are essentially \( \beta \)-integrable. For each \( n \), let \( h_{n} \) be a \( \beta \)-integrable function such that \( \phi_{A_{n}} = h_{n} \) locally \( \beta \)-a.e. Let \( B_{n} = \{x : h_{n}(x) = 1\} \). From \( \phi_{B_{n}} = h_{n} \phi_{B_{n}} \), we see that \( B_{n} \) is \( \beta \)-integrable. Since \( h_{n}^{2} = h_{n} \) locally \( \beta \)-a.e., it results that \( h_{n} = \phi_{B_{n}} \) locally \( \beta \)-a.e. (indeed, \( \beta \)-a.e. [2, §1, No. 3, Lemma 1]), therefore \( \phi_{A_{n}} = \phi_{B_{n}} \) locally \( \beta \)-a.e. It follows easily that \( S \subset B \cup N \), where \( B = \bigcup_{1}^{\infty} B_{n} \) is \( \beta \)-moderated and \( N \) is locally \( \beta \)-negligible; one can even suppose that \( B \) is a denumerable union of compact sets [2, §1, No. 2, Proposition 5]. Since \( N \) is also locally negligible for \( \mu = f \cdot \beta \) [2, §5, No. 5, Theorem 2], it follows that \( \mu \) is carried by \( B \); thus we are in the situation of Case (iii) of the Proposition.

**Remark.** In the Corollary, it also suffices that one of \( f, g \) be equal locally \( \beta \)-a.e. to a continuous function. More generally, suppose \( f, g, f', g' \) are locally \( \beta \)-integrable functions such that \( f = f' \) locally \( \beta \)-a.e. and \( g = g' \) locally \( \beta \)-a.e. It is elementary that if \( f \) and \( g \) are convolvable relative to \( \beta \), then so are \( f' \) and \( g' \), and one then has \( f \ast g = f' \ast g' \) locally \( \beta \)-a.e. Suppose, in addition, that \( f \ast g \) has a determination \( h \) such that \( h(x) \), for locally \( \beta \)-almost all \( x \), is given by the (coexisting) integral formulas of the Corollary; for such an \( x \), the first formula shows that \( f \) may be replaced by \( f' \) (in both formulas), the second that \( g \) may be replaced by \( g' \); thus \( f' \ast g' \) is also given by such formulas locally \( \beta \)-a.e.

**REFERENCES**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS, AUSTIN, TEXAS 78712