ABSTRACT. Let $T$ be an operator on a complex Hilbert space $H$. Some growth conditions on operator radius of the resolvent of $T$ are studied. Moreover, it is shown that the conjecture, due to V. Istrătescu, that for operators $T$ satisfying growth condition $(G_1)$

$$\sup_{\|x\|=1} \{\|Tx\|^2 - |(Tx, x)|^2\} = R_T^2,$$

where $R_T$ is the radius of the smallest circular disk containing the spectrum $\sigma(T)$, turns out to be false.

Let $T$ be an operator on a complex Hilbert space $H$. Let $\pi(T), \pi_0(T)$ and $W(T)$ denote the approximate point spectrum, point spectrum and the numerical range of $T$. We write $\lambda(T)$ and $|W(T)|$ for the spectral radius and the numerical radius of $T$, let $\text{Bdry} \sigma(T)$ and $\text{Conv} \sigma(T)$ denote the boundary and the convex hull of $\sigma(T)$.

An operator $T$ is called transloid if $T + zI$ is normaloid, that is, if

$$(T + zI) = \|T + zI\|$$

for all complex numbers $z$. If $\|T - zI\| = 1/d(z, \sigma(T))$, where $d(z, \sigma(T))$ denotes the distance of $\sigma(T)$ from the point $z$ (equivalently if $(T - zI)^{-1}$ is normaloid for all $z \not\in \sigma(T)$), then $T$ is said to satisfy the growth condition $(G_1)$ or $T$ is called an operator with $(G_1)$ property.

Let $C_\rho (\rho > 0)$ be the class of all operators with unitary $\rho$-dilation in the sense of [6]. According to Holbrook [3], an operator radius of $T$ is defined by

$$w_\rho(T) = \inf \{\alpha: \alpha > 0 \text{ and } \alpha^{-1}T \in C_\rho\}.$$

In particular, $w_1(T) = \|T\|$ and $w_2(T) = |W(T)|$. For further properties of operator radii, we refer to [3]. An operator $T$ is called $\rho$-oid if $w_\rho(T) = \lambda(T)$. Clearly, $1$-oid and $2$-oid operators are, respectively, the normaloid and spectralloid operators. According to [9], $T$ is called an operator of class $M_\rho (\rho \geq 1)$ if $w_\rho((T - zI)^{-1}) = 1/d(z, \sigma(T))$; equivalently if $(T - zI)^{-1}$ is $\rho$-oid for all $z \not\in \sigma(T)$. Clearly $M_1$ consists of all operators with the $(G_1)$ property, and since $M_\rho \subseteq M_{\rho'}$ for $\rho < \rho'$, it follows, in particular, that $M_1 \subseteq$
The fact that operators of class $M_p$ are convexoid follows immediately from the following criterion for convexoidity [9] which is an improved form of the well-known criterion due to Orland [8]: An operator $T$ is convexoid if and only if $w_p[(T - z)^{-1}] \leq 1/d(z, \sigma(T))$ for all $z \notin \sigma(T)$.

Recently, Lin [13, Theorem 5] has obtained the following improvement of Nieminen's result [7, Theorem 1]: $T$ is selfadjoint if and only if $w_p[(T - i\xi)^{-1}] \leq 1/|\xi|$ for every real $\xi \neq 0, 1 \leq p \leq 2$. Our first result shows that it is true even for $p > 2$.

**Theorem 1.** If $T$ is an operator such that the imaginary axis, with the possible exception of the origin, belongs to the resolvent set of $T$ and $w_p[(T - i\xi)^{-1}] \leq 1/|\xi|$ for every real $\xi \neq 0$, then $T$ is selfadjoint.

**Proof.** Since the given condition implies

$$w_p[(I + i\xi^{-1}T)^{-1}] = |\xi|w_p[(T - i\xi)^{-1}] \leq 1, \quad (I + i\xi^{-1}T)^{-1} \in C_p.$$  

Using [6, Theorem I.11.1] with $z = 1$ we obtain

$$(p - 2)\|(I - (I + i\xi^{-1}T)^{-1})x\|^2 + 2 \Re((I - (I + i\xi^{-1}T)^{-1})x, x) \geq 0$$

for all $x \in H$, that is,

$$(p - 2)\|i\xi^{-1}T(I + i\xi^{-1}T)^{-1}x\|^2 + 2 \Re(i\xi^{-1}T(I + i\xi^{-1}T)^{-1}x, x) \geq 0.$$  

Writing $(I + i\xi^{-1}T)^{-1}x = y$, we get

$$(p - 2)\|i\xi^{-1}Ty\|^2 + 2 \Re(i\xi^{-1}Ty, (I + i\xi^{-1}T)y) \geq 0$$

for all $y \in H$, that is

$$(p - 2)|\xi|^{-2}\|Ty\|^2 - 2\xi^{-1}\Im(Ty, y) + 2|\xi|^{-2}\|Ty\|^2 \geq 0,$$

or

$$(p/|\xi|^2)\|Ty\|^2 \geq 2\xi^{-1}\Im(Ty, y),$$

or

$$\rho\|Ty\|^2 \geq 2|\xi|^2\xi^{-1}\Im(Ty, y).$$

Now let $\xi = n \Im(Ty, y)$; we have

$$2n|\Im(Ty, y)|^2 \leq \rho\|Ty\|^2.$$  

Taking limit as $n \to \infty$, we get $\Im(Ty, y) = 0$, or $(Ty, y) = (y, Ty)$ for every $y \in H$. Therefore $T$ is selfadjoint. This completes the proof.

Next, we would like to ask whether the result of Donoghue [2] can be
improved along similar lines. More explicitly: Is $T$ unitary if $w_{\alpha}(T^{-1}) \leq 1$ and $w_{\rho}[(T - z)^{-1}] \leq 1/(|z| - 1)$, $|z| > 1$? The following theorem answers the question affirmatively.

**Theorem 2.** Let $T$ be an operator such that:
(i) $\sigma(T) \subseteq C$, where $C$ is the unit circle;
(ii) $w_{\alpha}(T^{-1}) \leq 1$, $\alpha \geq 1$; and
(iii) $w_{\rho}[(T - z)^{-1}] \leq 1/(|z| - 1)$, $1 < |z| < \delta < (\rho - 1)/(\rho - 2)$, $\rho > 2$.

Then $T$ is unitary.

**Proof.** First we observe that conditions (ii) and (iii) are equivalent to $T^{-1} \in C_\alpha$ and $(|z| - 1)(T - z)^{-1} \in C_{\rho}$ [3, Theorem 3.1], respectively. By Remark 3 of [6, Proposition 1.1.2], the latter condition yields
$$
\|[(T - z)^{-1} - \mu/(|z| - 1)]^{-1}\| \leq (|z| - 1)/(|\mu| - 1),
1 < |\mu| < (\rho - 1)/(\rho - 2).
$$

Since $|\mu|/(|z| - 1) > (|\mu| - 1)/(|z| - 1)$, Lemma 1 of [12] gives
$$
\|[(T - z - (|z| - 1)\mu)/(2|\mu| - 1)]^{-1}\| \leq (2|\mu| - 1)/(|\mu| - 1)(|z| - 1),
1 < |\mu| < (\rho - 1)/(\rho - 2).
$$

Setting $z = -\mu$, we obtain
$$
\|(T - u)^{-1}\| \leq 1/(|u| - 1), \quad 1 < |u| < k,
$$
where $u = -|\mu|\mu/(2|\mu| - 1)$ and $k$ is some constant. Therefore $T \in C_\beta$ for some $\beta > 2$. Since $T^{-1} \in C_\alpha$, by Corollary 4 of [12] it follows that $T$ is unitary. The proof is complete.

In particular, the following corollary includes Donoghue’s result [2].

**Corollary 1.** Let $T$ be an operator such that
(i) $\sigma(T) \subseteq C$,
(ii) $w_{\alpha}(T^{-1}) \leq 1$, $\alpha \geq 1$, and
(iii) $w_{\rho}[(T - z)^{-1}] \leq 1/(|z| - 1)$, $1 < |z| < \delta$, $1 \leq \rho \leq 2$.

Then $T$ is unitary.

**Proof.** Choose $\rho'$ such that $2 < \rho' < (2\delta - 1)/(\delta - 1)$. Then it follows from (iii) that $(|z| - 1)(T - z)^{-1} \in C_{\rho'} \subseteq C_{\rho}$, $1 < |z| < \delta < (\rho' - 1)/(\rho' - 2)$. Hence, the result follows from Theorem 2.

**Corollary 2.** If $T \in M_\rho$ and $\sigma(T) \subseteq C$, then $T$ is unitary.

**Proof.** Since $T \in M_\rho$,
$$
w_{\rho}[(T - z)^{-1}] \leq 1/d(z, \sigma(T)) \leq 1/(|z| - 1)
$$
for all $z \notin \sigma(T)$. In particular, we have $w_{\rho}(T^{-1}) \leq 1$ and $w_{\rho}[(T - z)^{-1}] \leq 1/(|z| - 1)$ for $|z| > 1$. Therefore the result follows from Theorem 2 and Corollary 1.
Our next theorem includes the following result implicitly proved by T. Saito [10, Theorem 1].

**Theorem A.** Let $T$ be an operator satisfying the growth condition (G) and let $\alpha_0$ be a point in $\sigma(T)$. If there exist sequences $\{z_n\}$ and $\{r_n\}$ such that

(i) $\{z:|z - z_n| < r_n\} \subset \text{complement of } \sigma(T)$,
(ii) $z_n \to \alpha_0$ and $|z_n - \alpha_0|/r_n \to 1$,

then

$$E(T - \alpha_0I) = E(T^* - \overline{\alpha}_0I),$$

where

$$E(T - \alpha_0I) = \{x: x \in H, \|x\| = 1 \text{ and } \|(T - \alpha_0I)x\| \to 0\}.$$ 

**Theorem 3.** Let $T$ be an operator of class $M\rho$. If $\alpha_0$ is a point of $\sigma(T)$ satisfying the conditions of Theorem A, then $E(T - \alpha_0I) = E(T^* - \overline{\alpha}_0I)$.

**Proof.** Changing the Hilbert space suitably and by a faithful $*$-representation $T \to T^0$, one can suppose $\pi(T) = \pi_0(T)$ and thereupon, $\text{Bdry } \sigma(T) \subset \pi(T) = \pi_0(T)$ [1]. Since $T \geq 0$ if and only if $T^0 \geq 0$, it is easily seen that $T \in C\rho$ if and only if $T^0 \in C\rho$. Consequently, $w\rho(T) = w\rho(T^0)$. To prove the assertion, it will suffice to show that $\ker(T - \alpha_0I) = \ker(T^* - \overline{\alpha}_0I)$.

Since $T - zI \in M\rho$ whenever $T \in M\rho$, we can assume that $\alpha_0 = 0$. Then $z_n \to 0$ and $|z_n|/r_n \to 1$. Let $\alpha_n = |z_n| - r_n$. Clearly, $\alpha_n/r_n \to 0$. Let $a_n = e^{i\arg z_n}$ and $T_n = (a_n \alpha_n - z_n)(T - z_n)^{-1}$. Then, by hypothesis,

$$w\rho(T_n) = |a_n \alpha_n - z_n| w\rho([T - z_n]^{-1})$$

$$\leq |a_n \alpha_n - z_n|/d(z_n, \sigma(T)) = |a_n |r_n|/d(z_n, \sigma(T)) \leq 1.$$

This shows that $T_n \in C\rho$ [3, Theorem 3.1]. Now we assert that

(a) $Tx = 0 \iff T_nx \to x$, and
(b) $T^*x = 0 \iff T^*_nx \to x$.

By symmetry, it suffices to prove (a). Assume that $Tx = 0$. Then

$$\|T_nx - x\| \leq \rho\|x - T_n^{-1}x\| = \rho\|x + (1/a_n r_n)(T - z_n)x\|$$

$$= \rho\|(I - |z_n|/r_n)x\| \to 0.$$

Thus $T_nx \to x$. On the other hand, if $T_nx \to x$, then

$$\|(T - a_n \alpha_n)x\| = \|(T - z_n) - (a_n \alpha_n - z_n)x\| = \|(T - z_n)(I - T_n)x\|$$

$$\leq \left(\|T\| + \sup_n |z_n|\right)\|T_nx - x\| \to 0.$$
\[ \|Tx\| = \lim \alpha_n \|x\| \leq (\sup r_n)(\lim(\|\alpha_n\|/r_n))\|x\| = 0. \]

Thus \( Tx = 0 \). Now to complete the proof it is enough to show that \( T_n x \to x \iff T_n^* x \to x \). Since \( T_n \in \mathcal{C}_\rho \), it follows from [6, Theorem 1.11.1] that

\[ (\rho - 2)\| (I - T_n) v \|^2 + 2\Re((I - T_n)v, v) \geq 0, \quad v \in \mathcal{H}. \]

Thus \( \Re S_n \geq 0 \) where \( S_n = (\rho - 2)(I - T_n)^*(I - T_n) + 2(I - T_n) \). Therefore we have

\[ \Re S_n \geq \Re S_n(x, y)^2 \leq \Re S_n(x, x)(\Re S_n(y, y), x, y \in \mathcal{H}. \]

Since \( T_n \in \mathcal{C}_\rho \) and, hence, \( \|T_n\| \leq \rho \), it follows that the sequence \( \{S_n\} \) is bounded; thus the last inequality reduces to

\[ \Re S_n \geq k_1 \|y\|^2 \Re S_n(x, x), \quad x, y \in \mathcal{H}, \]

for some constant \( k_1 \).

Now if \( T_n x \to x \), then \( S_n x \to 0 \). This, together with the above inequality, implies that \( \Re S_n \to 0 \), and so \( S_n x \to 0 \), or \( T_n^* x \to x \). A similar argument proves that \( T_n^* x \to x \Rightarrow T_n x \to x \). This completes the proof.

V. Istrătescu [4, Theorem 2] has proved that if \( T \) is transloid, then

\[ \sup \frac{1}{\|x\|} \Re S_n \geq R^2_T, \]

where \( R_T \) is the radius of the smallest circular disk containing \( \sigma(T) \). Moreover, he has conjectured that \((*)\) holds if \( T \) is convexoid or more particularly an operator with (G_1) property. Recently Sheth [11] has disproved the first part of the conjecture. Here we prove the falsity of the second part of the conjecture.

**Theorem 4.** There exists an operator satisfying the growth condition (G_1) for which \((*)\) is not true.

**Proof.** Consider the operator \( A = [0 1] \) on the two-dimensional Hilbert space \( \mathcal{H}_1 \). Let \( B \) be a normal operator on an infinite-dimensional Hilbert space \( \mathcal{H}_2 \) such that \( \sigma(B) = W(A) \). Write \( T = A \oplus B \). Then \( T \) satisfies the growth condition (G_1) [5, Theorem 1.2]. Let \( x = (0, 1) \oplus 0 \). Clearly, \( (Tx, x) = 0 \) and \( \|Tx\| = 1 \). Therefore

\[ \sup \frac{1}{\|x\|} \|Tx\|^2 - |(Tx, x)|^2 \geq 1. \]

Since \( \sigma(T) = W(A) \) is the disk with the center at the origin and radius \( 1/2 \), it follows that \( R_T = 1/2 \). Thus

\[ \sup \frac{1}{\|x\|} \|Tx\|^2 - |(Tx, x)|^2 > R^2_T. \]

This proves the result.
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