ARCS DEFINED BY ONE-PARAMETER SEMIGROUPS
OF OPERATORS IN BANACH SPACES WITH
THE RADON-NIKODYM PROPERTY

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ABSTRACT. It is shown that a recent theorem of Junghenn and Taam
concerning the domain of the infinitesimal generator of a strongly continuous
one-parameter semigroup of operators on a reflexive, locally convex
topological vector space remains valid if the domain of the operators is a
Banach space with the Radon-Nikodym property. A partial result is ob-
tained for general Banach spaces.

The following theorem is proved in a recent paper by Junghenn and
Taam:

Theorem. Let $X$ be a reflexive, locally convex topological vector
space. Let $T(t)$, for $t \geq 0$, be a strongly continuous semigroup of operators
on $X$, such that for all $c > 0$, $T(c)$ is an isomorphism (into), and let $A$
be the infinitesimal generator of the semigroup. The following are equivalent:

1. $x$ is in the domain of $A$;
2. $T(t)x$ is absolutely continuous on the interval $[0, c]$ for any $c > 0$;
3. $T(t)x$ is of bounded variation on the interval $[0, c]$ for any $c > 0$.

In the absence of reflexivity, each condition implies the next. It is the
purpose of this note to show that if $X$ is a Banach space, then (2) and (3)
are equivalent, and if, in addition, $X$ has the Radon-Nikodym property, then
all three are equivalent. Among spaces with the Radon-Nikodym property
are reflexive Banach spaces and separable dual Banach spaces. For a dis-
cussion of the Radon-Nikodym property, see the paper of Rieffel [2]. Recent
results characterizing Banach spaces with this property are to be found in
[4], [5], and [6].

To prove the above assertions we need two lemmas. Our notation is
that of [1]. Terms not defined in this note are as used in that paper.

Lemma 1. Let $X$ be a Banach space, and $x \in X$, for which (3) holds.
If $c > 0$ and $L$ is the total variation of $T(t)x$ on $[0, c]$, then
$\|T(c+h)x - T(c)x\|/h \leq KL/(c-h)$ for all $0 < h < c$, where $K$ is a constant
not dependent upon the choice of $h$ (but dependent upon $c$).
Proof. Choose \( n \) so that \( nh < c < (n + 1)h \), and let \( 1 \leq j \leq n \). Pick \( K \) so that \( \| T(t)x \| \leq K\| x \| \) for \( t \) in \([0, c]\). Then

\[
\| T(c + h)x - T(c)x \| = \| T(c + h - jh)T(jh)x - T(jh - h)x \| \\
\leq K\| T(jh)x - T(jh - h)x \|.
\]

Therefore

\[
n\| T(c + h)x - T(c)x \| \leq \sum_{j=1}^{n} K\| T(jh)x - T(jh - h)x \| \leq KL.
\]

The assertion follows since we have \( c - h < nh \).

Lemma 2. Under the same assumptions as Lemma 1, the set 
\( \{ \| T(h)x - x \| / h \mid 0 < h \leq c \} \) is bounded.

Proof. We have

\[
\| T(h)x - x \| / h = \| T(1) \left( T(c + h)x - T(c)x \right) \| / h \leq M\| T(c + h)x - T(c)x \| / h,
\]

for \( M = \| T(1)^{-1} \| \). Then \( \| T(h)x - x \| / h \leq MKL/(c - h) \), and if \( 0 < h < c/2 \) we have \( \| T(h)x - x \| / h \leq 2MKL/c = Q \). Clearly, for \( h \) in \([c/2, c]\), the numbers are bounded.

We prove now that (3) implies (2) in any Banach space. Let \( e > 0 \) be given and let \( (a_i, b_i), i = 1, 2, \ldots, n, \) be any collection of nonoverlapping intervals in \([0, c]\). If we first assume that \( b_i - a_i < c/2 \), then

\[
\| T(b_i)x - T(a_i)x \| = \| T(a_i)(T(b_i - a_i)x - x) \| \leq K\| T(b_i - a_i)x - x \|.
\]

Therefore the choice of \( d < e/2K \) gives \( \sum_{i=1}^{n} \| T(b_i)x - T(a_i)x \| < e \) if \( \sum_{i=1}^{n} (b_i - a_i) < d \).

Banach spaces \( X \) with the Radon-Nikodym property are precisely those spaces with the property that every function of bounded variation from the real line into \( X \) is differentiable almost everywhere. This is a classical result due to Bochner and Taylor [3]. If we assume that \( X \) has this property, then there is a \( T > 0 \) at which point the function \( T(\ )x \) is differentiable, if (3) holds. Then, as \( h \) approaches 0, \( [T(T + h)x - T(T)x]/h \) approaches a limit. It follows that \( [T(h)x - x]/h \) does also, since it is equal to \( [T(T)^{-1}] [T(T + h)x - T(T)x]/h] \). Therefore, if \( X \) is assumed to have the Radon-Nikodym property, the three conditions given in the Theorem are equivalent.

REFERENCES


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