CHARACTERIZING A CIRCLE
WITH THE DOUBLE MIDSET PROPERTY

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ABSTRACT. A short and elementary proof is given to show that a space \( X \) is a circle with the natural geodesic metric if \( X \) is a nondegenerate, complete, convex metric space with the double midset property.

In 1970 Berard [1] announced that a complete convex metric space having the double midset property must be a topological simple closed curve. Based on a manuscript [2] subsequently received from him, Loveland and Valentine [5] showed that under Berard's hypothesis the space is actually isometric to a circle having the natural geodesic metric. Although Berard's manuscript was not published, a paper by Berard and Nitka [3] has recently appeared in which the isometry is established. However a short and elementary proof can be obtained by quoting Theorem 1 of [6] and Theorem 2 of [5], with some extra work. Rather than proceed in this manner we have endeavored to make this note largely self-contained. Thus we adapt some proofs to our situation and give them here.

The midset (called the "bisector" in [3]) of two points \( a \) and \( b \) of a metric space \( X \) is the set of all \( x \) in \( X \) such that the distances \( ax \) and \( bx \) are equal, and \( X \) is said to have the double midset property (DMP) if, for every pair of distinct points \( a \) and \( b \) of \( X \), the midset \( M(a, b) \) of \( a \) and \( b \) consists of two points.

In the remainder of the paper, \( X \) will denote a nondegenerate, complete, convex, metric space having the DMP. It is easy to see that the "complete, convex" hypothesis can be replaced by "segment-convex" as is done in [3]. The essential hypothesis is that \( X \) contain with two of its points a segment joining them (see [4, Theorem 14.1, p. 41]).

Lemma 1. The space \( X \) contains a simple closed curve.

Proof. Let \( a \) and \( b \) be points of \( X \), \( S \) a segment with endpoints \( a \) and \( b \), and \( M(a, b) = \{m_1, m_2\} \). Since \( S \) cannot have two midpoints, \( m_2 \notin S \). Let \( S' \) be the union of two segments \( S_1 \) and \( S_2 \) having endpoints \( \{a, m_2\} \) and \( \{b, m_2\} \), respectively. Obviously \( S \cup S' \) contains a simple closed curve unless \( S_1 \) and \( S_2 \) share a segment \( S_3 \) with endpoints \( m_1 \) and \( m_2 \). However
Lemma 2 [6, Theorem 1]. The space \( X \) is a topological simple closed curve.

Proof. From Lemma 1, \( X \) contains a simple closed curve \( J \). Suppose there is a point \( x \) in \( X - J \). There cannot exist two points \( a \) and \( b \) of \( J \) equidistant from \( x \), for then \( M(a, b) \) would contain \( x \) and could intersect \( J \) at most once. This would contradict the fact that \( M(a, b) \) separates \( a \) from \( b \) in \( X \). Thus the function \( g: J \to R \), defined by \( g(t) = xt \), is a continuous injection of \( J \) into the real line. This is impossible since \( g \) is a homeomorphism under these conditions.

Main theorem [5, Theorem 2]. The space \( X \) is isometric to a circle having the natural geodesic metric.

Proof. Let \( a \) and \( x \) be two points of the simple closed curve \( X \) (see Lemma 2), and let \( S(a, b) \) be a maximal (with respect to inclusion) segment containing \( x \) and having \( a \) as an endpoint. Let \( \{x_i\} \) be a monoton sequence of points of the open arc \( X - S(a, b) \) converging to \( b \). Since \( S(a, b) \cap S(a, x_{\infty}) = \{a\} \), we see that the closure of \( \bigcup_{i=1}^{\infty} S(a, x_i) \) is a segment \( S'(a, b) \) and that \( X = S(a, b) \cup S'(a, b) \). Let \( C \) be a circle in \( E^2 \) of radius \( ab/\pi \) with the geodesic metric. Let \( f \) be a homeomorphism taking \( X \) onto \( C \) such that \( f|S(a, b) \) and \( f|S'(a, b) \) are both isometries onto semicircles of \( C \). To show that \( f \) is an isometry it suffices to check that \( f(x)f(y) = xy \) whenever \( x \) and \( y \) are chosen in the interiors of \( S(a, b) \) and \( S'(a, b) \), respectively. We denote \( f(z) \) by \( z' \), and we define \( pqr \) to mean \( pq + qr = pr \). We may assume that \( xay \) holds. If \( x'a'y' \) also holds, then \( xy = x'y' \) as desired. Otherwise we have \( x'b'y' \), and we now show this implies \( xby \) from which \( xy = x'y' \). Suppose \( xb + by > xy \). Since \( xy = xa + ay \), this leads to the contradiction that \( x'y' > x'a' + a'y' \).

REFERENCES

2. ———, Characterizations of metric spaces by the use of their midsets: One-spheres (Unpublished manuscript, 1–14).