APPLICATIONS OF CLUSTER SETS IN MINIMAL TOPOLOGICAL SPACES

T. R. HAMLETT

ABSTRACT. Given a function $f$ from a topological space $X$ into a topological space $Y$ and a point $x \in X$, the cluster set of $f$ at $x$ is $C(f; x) = \bigcap \{ Cl(f(U)) : U$ is a neighborhood of $x \}$, where $Cl(U)$ denotes the closure of $U$. In this paper, $Y$ is taken to be a minimal topological space and $C(f; x)$ is used as a tool to obtain information about the continuity of $f$.

1. Introduction. If $X$ is a topological space and $x \in X$, let $\mathcal{N}(x)$ denote the nbd (neighborhood) system at $x$. Given a function $f$ from a topological space $X$ into a topological space $Y$, J. D. Weston [6] defined the cluster set of $f$ at $x \in X$ to be

$$C(f; x) = \bigcap \{ Cl(f(U)) : U \in \mathcal{N}(x) \}$$

where $Cl(U)$ denotes the closure of $U$. Weston [6] observed that if $Y$ is a Hausdorff space and $f$ is continuous, then $C(f; x)$ is degenerate. He also noted that the converse holds provided $Y$ is compact, and, in general, does not hold if $Y$ is not compact. In this paper we take $Y$ to be either $H$-closed, minimal Hausdorff, minimal Urysohn, or minimal regular, and use $C(f; x)$ as a tool to obtain information about the continuity of $f$.

2. Preliminaries. In this section we give some basic definitions and establish two lemmas that are useful in the sections which follow.

Let $X$ be a topological space. An open subset $U$ of $X$ is said to be regular open [3, p. 92] if $U = \text{Int}(Cl(U))$, where Int denotes the interior operator. A subset $F$ of $X$ is said to be regular closed [3, p. 92] if $F = Cl(\text{Int}(F))$. If $f$ is a function from $X$ into a topological space $Y$, $f$ is said to be closed (almost closed) if $f(K)$ is closed in $Y$ for every closed (regular closed) set $K$ in $X$. We say $f$ is an open mapping if $f(V)$ is open in $Y$ for every open set $V$ in $X$.

**Lemma 1.** Let $f$ be a closed map from a regular space $X$ into a space $Y$. If $f^{-1}(y)$ is closed in $X$ for every $y \in Y$, then $C(f; x)$ is degenerate for every $x \in X$.
Proof. The proof follows easily from the observation $\mathcal{C}(f; x) = \bigcap \{ \mathcal{C}(\mathcal{U}); \mathcal{U} \in \mathcal{N}(x) \}$.

Lemma 2. Let $f$ be an almost closed injection from a Hausdorff space $X$ into a space $Y$, then $\mathcal{C}(f; x)$ is degenerate for every $x \in X$.

Proof. The proof follows easily from the well-known fact [3, p. 92] that the closure of an open set is regular closed and the observation $\mathcal{C}(f; x) \subseteq \bigcap \{ \mathcal{C}(\mathcal{U}); \mathcal{U} \in \mathcal{N}(x) \}$.

3. $H$-closed and minimal Hausdorff spaces. Let $f$ be a function from a space $X$ into a space $Y$. We say $f$ is almost continuous [4, Definition 3] at $x \in X$ if for every regular open nbd $V$ of $f(x)$, there exists a $W \in \mathcal{N}(x)$ such that $f(W) \subseteq V$.

Theorem 3.1. Let $f$ be an open mapping from a space $X$ into an $H$-closed space $Y$, and let $x \in X$. Then $\mathcal{C}(f; x)$ is degenerate if and only if $f$ is almost continuous at $x$.

Proof. Necessity. Let $V$ be a regular open nbd of $f(x)$. Suppose $f(U) \cap (Y - V)$ is nonempty for every nbd $U$ of $x$. Note that $(Y - V)$ is a regular closed subset of $Y$, and hence is $H$-closed [7, p. 127]. Now $\{ \mathcal{C}(f(U)) \cap (Y - V); U \in \mathcal{N}(x) \}$ is a family of closed sets, and their interiors with respect to $(Y - V)$ satisfy the finite intersection property. Thus $\mathcal{C}(f; x) \cap (Y - V)$ is nonempty and therefore $\mathcal{C}(f; x)$ is not degenerate.

Sufficiency. Assume $f$ is almost continuous at $x$ and let $\tau$ denote the topology on $Y$. Let $\tau_s$ denote the topology on $Y$ generated by the regular open sets of $\tau$. Now we have

$$\{f(x)\} = \bigcap \{ \mathcal{C}(f(U)); U \in \mathcal{N}(x) \} \supseteq \bigcap \{ \mathcal{C}(\mathcal{U}); U \in \mathcal{N}(x) \} \supseteq \mathcal{C}(f; x).$$

Hence $\mathcal{C}(f; x) = \{f(x)\}$ is degenerate.

Before stating the next theorem, we should point out that an example of an $H$-closed Urysohn space which is not compact may be found in [1, Example 3.13]. A function $f$ from a space $X$ into a space $Y$ is said to be connected [5] if $f(C)$ is connected in $Y$ for every connected subset $C$ of $X$.

Theorem 3.2. If $f$ is an open connected mapping from a locally connected space $X$ into an $H$-closed Urysohn space $(Y, \tau)$, then $f$ is almost continuous at $x \in X$ if and only if $\mathcal{C}(f; x)$ is countable.

Proof. Sufficiency. Assume $\mathcal{C}(f; x)$ is countable. Let $\mathcal{C}(x)$ be a nbd base of open connected sets at $x$. Note that $\mathcal{C}(\mathcal{U})$ is a regular closed connected set for every $U \in \mathcal{C}(x)$. Let $\tau_s$ denote the topology on $Y$ generated by the regular open sets of $\tau$. By Theorem 3.4(b) of [1], $(Y, \tau_s)$ is compact. Thus $\{ \mathcal{C}(f(U)); U \in \mathcal{C}(x) \}$ is a collection of $\tau_s$ continua directed
by inclusion, which implies $\mathcal{C}(f; x)$ is a $\tau_x$ continuum [7, Theorem 28.2, p. 203]. Consequently, $\mathcal{C}(f; x)$ must be either a single point or uncountable. Our assumption that $\mathcal{C}(f; x)$ is countable implies $\mathcal{C}(f; x)$ is a single point and, therefore, $f$ is almost continuous by Theorem 3.1.

Necessity. Theorem 3.1.

We now focus our attention on minimal Hausdorff spaces, and at the end of this section state a theorem which gives results for $H$-closed and minimal Hausdorff spaces in combined form.

Theorem 3.3. Let $f$ be an open mapping from a space $X$ into a minimal Hausdorff space $Y$. Then $f$ is continuous at $x \in X$ if and only if $\mathcal{C}(f; x)$ is degenerate.

Proof. We have only to show sufficiency. Assume $\mathcal{C}(f; x)$ is degenerate, and let $f(\mathcal{U}) = \{f(U): U \in \mathcal{U}(x) \text{ and } U \text{ is open}\}$. Now $f(\mathcal{U})$ is an open filterbase, and the assumption $\mathcal{C}(f; x)$ is degenerate implies $f(\mathcal{U})$ has a unique adherent point which must be $f(x)$. Since $Y$ is minimal Hausdorff, $f(\mathcal{U})$ converges to $f(x)$ and therefore $f$ is continuous.

We are now ready to apply Lemmas 1 and 2.

Theorem 3.4. Let $f$ be an open mapping from a space $X$ into a minimal Hausdorff (H-closed) space $Y$.

1. If $X$ is regular, $f$ is closed, and $f^{-1}(y)$ is closed for every $y \in Y$, then $f$ is continuous (almost continuous).

2. If $f$ is an almost closed injection and $X$ is Hausdorff, then $f$ is continuous (almost continuous).

Proof. (1) Lemma 1 and Theorem 3.3 (Theorem 3.1).

(2) Lemma 2 and Theorem 3.3 (Theorem 3.1).

Corollary 3.5. Every continuous bijection from an $H$-closed space onto a Hausdorff space has an almost continuous inverse.

Proof. The map $f^{-1}$ is an open closed injection from a Hausdorff space into an $H$-closed space. Now apply Theorem 3.4(2).

4. Minimal Urysohn and minimal regular spaces. In order to be consistent with [1], our definition of regular (in this section only) includes the $T_1$ separation property. An open filterbase $\mathcal{B}$ in a space is called a Urysohn filterbase if for each point $p \notin \bigcap\{\text{Cl}(B): B \in \mathcal{B}\}$, there is an open nbhd $U$ of $p$ and $B \in \mathcal{B}$ such that $\text{Cl}(U) \cap \text{Cl}(B) = \emptyset$. An open filterbase $\mathcal{U}$ is called a regular filterbase if for each $U \in \mathcal{U}$ there exists a $V \in \mathcal{U}$ such that $\text{Cl}(V) \subseteq U$. Note that a regular filterbase is a Urysohn filterbase. The following theorem gives a filterbase characterization of minimal regular and minimal Urysohn spaces.
Theorem 1.1 [1]. A regular (Urysohn) space is minimal regular (Urysohn) if and only if every regular (Urysohn) filterbase with a unique adherent point converges.

Theorem 4.2. Let \( f \) be an open and closed mapping from a regular space \( X \) into a minimal regular (Urysohn) space \( Y \). Then \( f \) is continuous at \( x \in X \) if and only if \( \mathfrak{O}(f; x) \) is degenerate.

Proof. We need only show sufficiency. Observe that for \( x \in X \), \( \{f(U) : U \in \mathfrak{N}(x)\} \) is a regular filterbase (and hence a Urysohn filterbase) and apply Theorem 4.1.

We are now ready for another application of Lemma 1.

Theorem 4.3. Let \( f \) be an open and closed mapping from a regular space \( X \) into a minimal regular (Urysohn) space \( Y \). Then \( f \) is continuous if and only if the preimages of points in \( Y \) are closed in \( X \).

Proof. Necessity is obvious and sufficiency follows from Lemma 1 and Theorem 4.2.

Acknowledgement. The author is indebted to his major professor, Paul E. Long, for his patience, guidance, and constructive criticism.

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ARKANSAS, FAYETTEVILLE, ARKANSAS 72701