NORMAL SUBGROUPS CONTAINED IN THE FRATTINI SUBGROUP. II

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ABSTRACT. If \( p \) is an odd prime and \( H \) is a \( p \)-group with a characteristic subgroup \( K \) such that \( |K| > |K \cap Z(H)| = p \), then \( H \) cannot be a normal subgroup contained in the Frattini subgroup of any finite group \( G \).

We consider only finite groups. The order of the group \( G \) is \(|G|\), \( Z(G) \) is the center of \( G \), \( A(G) \) is the automorphism group of \( G \) and \( I(G) \) is the group of inner automorphisms. If \( G \) is nilpotent, \( cl(G) \) denotes its nilpotence class. Other notation is also standard.

Our aim is to prove the following

**Theorem.** Let \( H \) be a \( p \)-group, \( p \) an odd prime, with a characteristic subgroup \( K \) such that \( |K| > |K \cap Z(H)| = p \). Then \( H \) cannot be a normal subgroup contained in the Frattini subgroup of any finite group \( G \).

This result appears in [6] for arbitrary prime \( p \), but under the additional hypothesis that \( cl(K) \neq 2 \). It appears in [3] for the case that \( p \) is any prime and \( G \) is \( p \)-supersolvable. The case that \( |H| = |K| = p^3 \) is covered in [5].

With no loss of generality (see [6]), we take \( K = H \) and \( cl(H) = 2 \). Then \( H \) is extra-special. For a discussion of extra-special \( p \)-groups and their automorphisms the reader is referred to [1], [7], and [8].

Our argument is based on two lemmas, the first of which is mentioned in [2]. (The author is grateful to Professor David Goldschmidt for a very helpful conversation concerning this result.)

**Lemma 1.** If \( H \) is an extra-special \( p \)-group of exponent \( p \), \( p \) odd, then \( A(H) \) splits over \( I(H) \).

**Proof.** \( H = \langle x_1, x_2, \ldots, x_n, z \rangle \) with \( x_i^p = z^p = 1 \) for each \( i \) and \( [x_1, x_2] = [x_3, x_4] = \cdots = [x_{n-1}, x_n] = z \). Further, \( [x_i, x_j] = 1 \) unless \( \{i, j\} \) is one of \( \{1, 2\}, \{1, 3\}, \{1, 4\}, \ldots, \{n-1, n\} \). Each element of \( H \) has unique representation as \( (\prod_{i=1}^n x_i^{a_i})z^b \) with \( 0 \leq a_i, b < p \).

If \( \sigma \in A(H) \), then for each \( i \), \( \sigma(x_i) = (\prod_{j=1}^n x_j^{a_{ij}})z^{b_i} \) with \( (a_{ij}) \in GL(n, p) \) and \( 0 \leq b_i < p \). Further, \( \sigma \in I(H) \) if and only if \( (a_{ij}) \) is the identity matrix.

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Now the mapping \( r \) of \( \{x_1, x_2, \ldots, x_n, z\} \) into \( H \), defined by \( r(x_i) = x_i^{-1} \) (\( i = 1, 2, \ldots, n \)) and \( r(z) = z \), determines an automorphism \( r \in \text{Aut}(H) \), and \( C_A(H)(r) \) has trivial intersection with \( l(H) \). Let \( \sigma \) map \( H \) into \( H \) and \( \gamma \) map \( \{x_1, x_2, \ldots, x_n, z\} \) into \( H \) and suppose that for \( i = 1, 2, \ldots, n \),
\[
\sigma(x_i) = \left( \prod_{j=1}^{n} x_i^{a_{ij}} \right) z^{b_i} \quad \text{and} \quad \gamma(x_i) = \left( \prod_{j=1}^{n} x_i^{a_{ij}} \right) z^{c_i}.
\]
Consider the system of linear congruences
\[
\sum_{j=1}^{n} a_{ij} j = c_i - b_i \quad (\text{mod } p), \quad i = 1, 2, \ldots, n.
\]
If \( (a_{ij}) \) is nonsingular, there exists a unique solution \( (d_1, d_2, \ldots, d_n) \) with \( 0 \leq d_i < p \). The mapping \( \rho \) of \( \{x_1, x_2, \ldots, x_n, z\} \) into \( H \) defined by \( \rho(x_i) = x_i^{d_i} \) (\( i = 1, 2, \ldots, n \)) and \( \rho(z) = z \) determines an inner automorphism \( \rho \in l(H) \) and
\[
\rho \sigma(x_i) = \rho \left( \left( \prod_{j=1}^{n} x_i^{a_{ij}} \right) z^{b_i} \right) = \left( \prod_{j=1}^{n} x_i^{a_{ij}} \right) z^{e_i} = \left( \prod_{j=1}^{n} x_i^{a_{ij}} \right) z^{c_i} = \gamma(x_i)
\]
where \( e_i = \sum_{j=1}^{n} a_{ij} d_j + b_i \). In particular, if \( \sigma \in \text{Aut}(H) \), then \( \gamma \) agrees with \( \rho \sigma \) on the generating set \( \{x_1, x_2, \ldots, x_n\} \) and, hence, determines an automorphism \( \gamma \in \text{Aut}(H) \) with \( \rho \gamma = \gamma \).

We now show that for arbitrary \( \tau \in \text{Aut}(H) \), the exponents \( c_i \) (\( i = 1, 2, \ldots, n \)) above can be selected so that \( \gamma \in C_A(H)(\tau) \). For \( i = 1, 2, \ldots, n \) let \( c_i \) be the unique solution of the linear congruence
\[
2t + f_i = 2t + \sum_{k=1}^{n-1} a_{ik} a_{i(k+1)} \equiv 0 \quad (\text{mod } p).
\]
Then
\[
\gamma \tau(x_i) = \gamma(x_i^{-1}) = \left( \prod_{j=1}^{n} x_i^{a_{ij}} \right)^{-1} z^{-c_i} = \left( \prod_{j=1}^{n} x_i^{-a_{ij}} \right) z^{-c_i - f_i} = \left( \prod_{j=1}^{n} x_i^{-a_{ij}} \right) z^{c_i} = \tau \left( \left( \prod_{j=1}^{n} x_i^{a_{ij}} \right) z^{c_i} \right) = \tau(y(x_i)).
\]
Thus, for each \( \tau \in \text{Aut}(H) \), there exists \( \rho \in l(H) \) and \( \gamma \in C_A(H)(\tau) \) such that \( \sigma = \rho^{-1} \gamma \), i.e. \( \text{Aut}(H) = l(H) C_A(H)(\tau) \). Hence, \( C_A(H)(\tau) \) complements \( l(H) \) in \( \text{Aut}(H) \), completing the proof of Lemma 1.

**Lemma 2.** If \( H \) is an extra-special \( p \)-group of exponent \( p^2 \), \( p \) odd, then \( H \) has a characteristic subgroup \( K \) of order \( p^2 \).

**Proof.** \( H = \langle x_1, x_2, \ldots, x_n, z \rangle \) with \( x_i^{p^2} = z^p = 1 \) (\( i = 2, 3, \ldots, n \)), \( x_1^{p^2} = z \) and \( [x_1, x_2] = [x_3, x_4] = \cdots = [x_{n-1}, x_n] = z \). Further, \( [x_i, x_j] = 1 \) unless \( \{i, j\} \) is one of \( \{1, 2\}, \{3, 4\}, \ldots, \{n-1, n\} \). The subgroup \( \langle z, x_2, \ldots, x_n \rangle \), consisting precisely of those elements in \( H \) satisfying \( x_i^p = 1 \), is characteristic in \( H \), and its center, also characteristic in \( H \), is...
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(z, x₂), of order p². Take K = ⟨z, x₂⟩. This proves Lemma 2.

Now, let H be an extra-special p-group, p odd. If the exponent of H is p, then Lemma 1 together with III. 3.2 and III. 3.13 of [7] implies that H cannot be a normal subgroup contained in the Frattini subgroup of any finite group G. If on the other hand the exponent of H is p², then H has a characteristic subgroup K of order p² (Lemma 2), which of necessity intersects Z(H) in a subgroup of order p. By [6], the desired conclusion follows, and the proof of the theorem is complete.

For the case p = 2, we have very little information. Again we lose no generality by taking H = K and cl(H) = 2. Thus, as before, H is extra-special. From (2), the splitting of A(H) over l(H) occurs for extra-special 2-groups of orders 2³ and 2⁵ and does not occur for those of order 2⁷ and larger. Hence, a 2-group H with characteristic subgroup K of order 2³ or 2⁵ and intersecting Z(H) in a subgroup of order 2 cannot be a normal subgroup contained in the Frattini subgroup of any finite group G. Since the splitting of A(H) over l(H) is only a sufficient condition for the above nonembeddability conclusion, the question remains open for extra-special 2-groups of larger orders.

Added in proof. Professor Homer Bechtell has observed that Griess' work (2) can be used to show that if H is an extra-special 2-group of order larger than 32, there exists a (nonsolvable) group G having Frattini subgroup H.

REFERENCES


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