NORMAL SUBGROUPS CONTAINED IN THE FRATTINI SUBGROUP. II

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ABSTRACT. If $p$ is an odd prime and $H$ is a $p$-group with a characteristic subgroup $K$ such that $|K| > |K \cap Z(H)| = p$, then $H$ cannot be a normal subgroup contained in the Frattini subgroup of any finite group $G$.

We consider only finite groups. The order of the group $G$ is $|G|$, $Z(G)$ is the center of $G$, $A(G)$ is the automorphism group of $G$ and $I(G)$ is the group of inner automorphisms. If $G$ is nilpotent, $cl(G)$ denotes its nilpotence class. Other notation is also standard.

Our aim is to prove the following

**Theorem.** Let $H$ be a $p$-group, $p$ an odd prime, with a characteristic subgroup $K$ such that $|K| > |K \cap Z(H)| = p$. Then $H$ cannot be a normal subgroup contained in the Frattini subgroup of any finite group $G$.

This result appears in [6] for arbitrary prime $p$, but under the additional hypothesis that $cl(K) \neq 2$. It appears in [3] for the case that $p$ is any prime and $G$ is $p$-supersolvable. The case that $|H| = |K| = p^3$ is covered in [5].

With no loss of generality (see [6]), we take $K = H$ and $cl(H) = 2$. Then $H$ is extra-special. For a discussion of extra-special $p$-groups and their automorphisms the reader is referred to [1], [7], and [8].

Our argument is based on two lemmas, the first of which is mentioned in [2]. (The author is grateful to Professor David Goldschmidt for a very helpful conversation concerning this result.)

**Lemma 1.** If $H$ is an extra-special $p$-group of exponent $p$, $p$ odd, then $A(H)$ splits over $I(H)$.

**Proof.** $H = \langle x_1, x_2, \ldots, x_n, z \rangle$ with $x_i^p = z^p = 1$ for each $i$ and $[x_1, x_2] = [x_3, x_4] = \cdots = [x_{n-1}, x_n] = z$. Further, $[x_i, x_j] = 1$ unless $|i - j|$ is one of $1, 2, 3, 4, \ldots, |n - 1, n|$. Each element of $H$ has unique representation as $(\prod_{i=1}^n x_i^{a_{ii}})z^b$ with $0 \leq a_{ii} < p$. If $x_i \in A(H)$, then for each $i$, $s_i(x_i) = (\prod_{j=1}^n x_j^{a_{ij}})z^b$ with $a_{ij} \in GL(n, p)$ and $0 \leq b_i < p$. Further, $x_i \in I(H)$ if and only if $a_{ij}$ is the identity matrix.
Now the mapping $\tau$ of $\{x_1, x_2, \ldots, x_n, z\}$ into $H$, defined by $\tau(x_i) = x_i^{-1}$ $(i = 1, 2, \ldots, n)$ and $\tau(z) = z$, determines an automorphism $\tau \in A(H)$, and $C_A(H)(\tau)$ has trivial intersection with $l(H)$. Let $\sigma$ map $H$ into $H$ and $\gamma$ map $\{x_1, x_2, \ldots, x_n\}$ into $H$ and suppose that for $i = 1, 2, \ldots, n$,

$$\sigma(x_i) = \left(\prod_{j=1}^{n} x_{ij}\right)^i z^i \quad \text{and} \quad \gamma(x_i) = \left(\prod_{j=1}^{n} x_{ij}\right)^j z^i.$$ 

Consider the system of linear congruences

$$\sum_{j=1}^{n} a_{ij} x_i = c_i - b_i \pmod{p}, \quad i = 1, 2, \ldots, n.$$ 

If $(a_{ij})$ is nonsingular, there exists a unique solution $(d_1, d_2, \ldots, d_n)$ with $0 \leq d_i < p$. The mapping $\rho$ of $\{x_1, x_2, \ldots, x_n, z\}$ into $H$ defined by $\rho(x_i) = x_i^{d_i}$ $(i = 1, 2, \ldots, n)$ and $\rho(z) = z$ determines an inner automorphism $\rho \in l(H)$ and

$$\rho \sigma(x_i) = \rho \left(\left(\prod_{j=1}^{n} x_{ij}\right)^i z^i \right) = \left(\prod_{j=1}^{n} x_{ij}\right)^i z^i = \gamma(x_i)$$

where $e_i = \sum_{j=1}^{n} a_{ij} d_i + b_i$. In particular, if $\sigma \in A(H)$, then $\gamma$ agrees with $\rho \sigma$ on the generating set $\{x_1, x_2, \ldots, x_n\}$ and, hence, determines an automorphism $\gamma \in A(H)$ with $\rho \sigma = \gamma$.

We now show that for arbitrary $\tau \in A(H)$, the exponents $c_i$ $(i = 1, 2, \ldots, n)$ above can be selected so that $\gamma \in C_A(H)(\tau)$. For $i = 1, 2, \ldots, n$ let $c_i$ be the unique solution of the linear congruence

$$2t + f_i = 2t + \sum_{k=1}^{n-1} a_{ik} a_{i(k+1)} \equiv 0 \pmod{p}.$$ 

Then

$$\gamma \tau(x_i) = \gamma(x_i^{-1}) = \left(\prod_{j=1}^{n} x_{ij}\right)^{-1} z^{-c_i} = \left(\prod_{j=1}^{n} x_{ij}^{-1}\right)^{-c_i} = \left[\prod_{j=1}^{n} x_{ij}^{-1}\right]^{-c_i}.$$ 

Thus, for each $\tau \in A(H)$, there exists $\rho \in l(H)$ and $\gamma \in C_A(H)(\tau)$ such that $\sigma = \rho^{-1} \gamma$, i.e., $A(H) = l(H)C_A(H)(\tau)$. Hence, $C_A(H)(\tau)$ complements $l(H)$ in $A(H)$, completing the proof of Lemma 1.

**Lemma 2.** If $H$ is an extra-special $p$-group of exponent $p^2$, $p$ odd, then $H$ has a characteristic subgroup $K$ of order $p^2$.

**Proof.** $H = \langle x_1, x_2, \ldots, x_n, z \rangle$ with $x_i^p = x_i = z$ and $[x_i, x_j] = [x_3, x_4] = \cdots = [x_{n-1}, x_n] = z$. Further, $[x_i, x_j] = 1$ unless $\{i, j\}$ is one of $\{1, 2\}, \{3, 4\}, \ldots, \{n - 1, n\}$. The subgroup $\langle z, x_2, \ldots, x_n \rangle$, consisting precisely of those elements in $H$ satisfying $x_i^p = 1$, is characteristic in $H$, and its center, also characteristic in $H$, is
\((z, x_2)\), of order \(p^2\). Take \(K = \langle z, x_2 \rangle\). This proves Lemma 2.

Now, let \(H\) be an extra-special \(p\)-group, \(p\) odd. If the exponent of \(H\) is \(p\), then Lemma 1 together with III. 3.2 and III. 3.13 of [7] implies that \(H\) cannot be a normal subgroup contained in the Frattini subgroup of any finite group \(G\). If on the other hand the exponent of \(H\) is \(p^2\), then \(H\) has a characteristic subgroup \(K\) of order \(p^2\) (Lemma 2), which of necessity intersects \(Z(H)\) in a subgroup of order \(p\). By [6], the desired conclusion follows, and the proof of the theorem is complete.

For the case \(p = 2\), we have very little information. Again we lose no generality by taking \(H = K\) and \(\text{cl}(H) = 2\). Thus, as before, \(H\) is extra-special. From (2), the splitting of \(A(H)\) over \(I(H)\) occurs for extra-special \(2\)-groups of orders \(2^3\) and \(2^5\) and does not occur for those of order \(2^7\) and larger. Hence, a \(2\)-group \(H\) with characteristic subgroup \(K\) of order \(2^3\) or \(2^5\) and intersecting \(Z(H)\) in a subgroup of order \(2\) cannot be a normal subgroup contained in the Frattini subgroup of any finite group \(G\). Since the splitting of \(A(H)\) over \(I(H)\) is only a sufficient condition for the above nonembeddability conclusion, the question remains open for extra-special \(2\)-groups of larger orders.

Added in proof. Professor Homer Bechtell has observed that Griess' work (2) can be used to show that if \(H\) is an extra-special \(2\)-group of order larger than \(32\), there exists a (nonsolvable) group \(G\) having Frattini subgroup \(H\).

REFERENCES