ON LAKSHMIKANTHAM’S COMPARISON
FOR ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. The paper describes the relation between Lakshmikantham’s comparison and other known facts.

V. Lakshmikantham introduced in [4] a new setting for the comparison of ordinary differential equations. The aim of this paper is to show that Lakshmikantham's comparison is a special case, in the sense explained by the statement of the theorem below, of the pairing comparison considered in Vidossich [6]. The implications of this discovery are: (i) all the results in the framework of Lakshmikantham's pairing can be generalized by setting them in the framework of the pairing comparison; (ii) the stability theorem Lakshmikantham and Leela [5, Theorem 2.13.1] claimed as new is in reality a special case of an older one [5, Theorem 2.13.3] which appeared in Conti and Sansone [2].

The above claims are consequences of the theorem below. Concerning (i), we avoid carrying out the program since the interested reader can do it easily; the proofs of the generalizations can be based on the existing arguments of the theorems to be generalized simply by substituting $\|\cdot\|$ by $\|\cdot\|^2$. We refer to Lakshmikantham and Leela [5], Ladas and Lakshmikantham [3] and Becker and Vidossich [1, Theorem 4] for the related results and bibliography.

Theorem. Let $X$ be a Banach space, $A \subseteq \mathbb{R} \times X$, $B \subseteq \mathbb{R}^2$ and $f, g : A \to X$, $\omega : B \to \mathbb{R}$ continuous functions. Let $O : \mathbb{R} \to \mathbb{R}$ be such that

$$\lim_{h \to 0} \frac{O(h)}{h} = 0.$$

Then each one of the following statements,

(1) $\|x + hf(t, x)\| \leq \|x\| + h\omega(t, \|x\|) + O(h),$  

(2) $\|x - y + hf(t, x) - f(t, y)\| \leq \|x - y\| + h\omega(t, \|x - y\|) + O(h),$  

(3) $\|x - y + hf(t, x) - g(t, y)\| \leq \|x - y\| + h\omega(t, \|x - y\|) + O(h),$  

implies the corresponding one of the following statements,

(1)* $(f(t, x), x)_{-} \leq \omega(t, \|x\| \|x\|),$  

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(2) \[ (f(t, x) - f(t, y), x - y) \leq \omega(t, \|x - y\|)\|x - y\|, \]

(3) \[ (f(t, x) - g(t, y), x - y) \leq \omega(t, \|x - y\|)\|x - y\|, \]

but the converse fails. In other words, we have

\[(i) \Rightarrow (i)^* \not\Rightarrow (i) \quad (i = 1, 2, 3).\]

In the above statement, \((\cdot, \cdot)\) denotes the generalized inner product on a Banach space \(X\),

\[(x, y)_\cdot = \inf\{h(x)| h \in J(y)\},\]

where \(J: X \to 2^*\) is the duality map

\[J(x) = \{h \in X^*|\|h\| = \|x\|, h(x) = \|x\|^2\}.\]

When \(X\) is a Hilbert space, \((\cdot, \cdot)_\cdot\) coincides with the inner product.

**Proof of Theorem.** First we note the following property of the generalized pairing

\[(x + y, x)_\cdot = \|x\|^2 + (y, x)_\cdot.\]

For, choosing \(h \in J(x)\), we have

\[h(x + y) = h(x) + h(y) = \|x\|^2 + h(y)\]

from which we have \((*)\) by taking \(\inf_h\).

\[(1) \Rightarrow (1)^*. \text{ From (1) it follows that}\]

\[b\omega(t, \|x\|) + O(h) \geq \|x + hf(t, x)\| - \|x\|.\]

We have

\[b\omega(t, \|x\|)\|x\| + O(h)\|x\| \geq \|x + hf(t, x)\|\|x\| - \|x\|^2\]

(by the preceding inequality)

\[\geq (x + hf(t, x), x)_\cdot - \|x\|^2\]

(by Cauchy-Schwartz inequality)

\[\geq \|x\|^2 + (hf(t, x), x)_\cdot - \|x\|^2 \quad \text{(by \((*)\))}\]

\[= hf(t, x, x)_\cdot.\]

Dividing both members of this inequality by \(b > 0\) and taking \(\lim_{b \to 0}\) we get \((1)^*\).

\[(2) \Rightarrow (2)^* \text{ and } (3) \Rightarrow (3)^* \text{ can be proved as } (1) \Rightarrow (1)^*. \text{ The other part of the}\]

\[\text{Theorem will be proved by exhibiting counterexamples in the real line.}\]

\[(1)^* \not\Rightarrow (1). \text{ Let } X = \mathbb{R} \text{ and define } f, \omega: \mathbb{R} \times ]0, 1[ \to \mathbb{R} \text{ by}\]

\[f(t, x) = \sin x^{-1} - \omega(t, x).\]

Then \((1)^*\) holds for \(f, \omega). \text{ Assume (1) holds for some function } O \text{ with}\]

\[\lim_{b \to 0} (O(h)/h) = 0\] and argue for a contradiction. Fix \(h \in ]0, 1[\). There is a sequence \((x_n)\) in \([0, h]\) such that \(\lim_{n \to \infty} x_n = 0\) and \(\sin(1/x_n) = -1. \text{ We have}\]

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Therefore from the assumed (1) we have
\[ h - x_{h,n} \leq x_{h,n} - h + O(h). \]
Taking \( \lim_n \) we get \( h \leq -h + O(h) \) which implies \( O(h)/h \geq 2 \) (all \( h \)), a contradiction.

(2) \( \nRightarrow \) (2). Let \( X = \mathbb{R} \) and define \( f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}, \omega : \mathbb{R}^2 \to \mathbb{R} \) by
\[ f(t, x) = -tx, \quad \omega(t, x) = 0. \]
Since \( f(t, \cdot) \) is decreasing, (2)* holds for \( f, \omega \). Assume (2) holds for some function \( \varphi \) with \( \lim_{h \to 0} (\varphi(h)/h) = 0 \), and argue for a contradiction. Fix \( h > 0 \).

There are sequences \( (x_{h,n})_n, (y_{h,n})_n \) of real numbers such that
\[ 0 < x_{h,n} - y_{h,n} < h \quad \text{and} \quad \lim_n (x_{h,n} - y_{h,n}) = 0. \]
Let \( t_{h,n} = 1/(x_{h,n} - y_{h,n}) \). We have
\[ |x_{h,n} - y_{h,n} + h(f(t_{h,n}, x_{h,n}) - f(t_{h,n}, y_{h,n}))| = h - (x_{h,n} - y_{h,n}), \]
\[ |x_{h,n} - y_{h,n}| + h\omega(t_{h,n}, |x_{h,n} - y_{h,n}|) + O(h) = x_{h,n} - y_{h,n} + O(h). \]
Therefore from the assumed (2) we have
\[ h - (x_{h,n} - y_{h,n}) \leq x_{h,n} - y_{h,n} + O(h). \]
Taking \( \lim_n \) we get \( h \leq O(h) \) which implies \( O(h)/h \geq 1 \) (all \( h \)), a contradiction.

(3) \( \nRightarrow \) (3). This follows by taking \( g = f \) in the above example used to show (2)* \( \nRightarrow \) (2). q.e.d.

REFERENCES