CYLINDRICITY OF ISOMETRIC IMMERSIONS
INTO EUCLIDEAN SPACE

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ABSTRACT. A simple geometric proof is given for the Hartman-Nirenberg cylindricity theorem and some generalizations. Then the following cylindricity theorem (unpublished) of S. Alexander is proved using the same idea.

Theorem. Let $f: M \to E^n$ be an isometric Euclidean immersion of the Riemannian product $M = M_1 \times \ldots \times M_k \times E^m$ where the M_i are not everywhere flat Riemannian manifolds, and E^m denotes Euclidean m-space. Then $C \geq k$, where C denotes the codimension of the immersion; and if $C = k$, then the immersion is cylindrical on the Euclidean factor.

1. **Introduction.** In this paper we prove some cylindricity theorems for isometric immersions of Riemannian manifolds in Euclidean space. These theorems all depend upon a remarkable lemma of P. Hartman. In its simplest form the lemma states that an isometric immersion of Euclidean 2-space E^2 in E^3, which maps a straight line in E^2 onto a straight line in E^3, must be cylindrical.

We first give a short and amusing geometrical proof of Hartman's lemma, which remarkably enough shows that smoothness assumptions on the immersion can be essentially dispensed with. The method used is completely general and may be of use elsewhere. In any case it leads to what must be the simplest possible proof of the well-known Hartman-Nirenberg cylindricity theorem. In fact, following Hartman, we state as Theorem 1 a partial generalization to Euclidean immersions of spaces of nonnegative Ricci curvature.

Finally, in §3, we use Hartman's lemma to give a simple proof of an important unpublished cylindricity theorem of S. B. Alexander.

Although we have shown how to relax differentiability assumptions in the global part of these cylindricity theorems, we are not especially interested in that question here, so all maps and manifolds are assumed of class C^∞ except where otherwise indicated.

2. **The tetherball principle.** In this section we formulate a completely general geometric principle and use it to prove the lemma below of P. Hartman [4]. The lemma will be used to obtain cylindricity theorems in this section and the following.

In order to state the lemma without differentiability assumptions, we will need the following definition. Let M denote an arbitrary Riemannian manifold.

Definition. A map $f\colon M \to \mathbf{E}^N$ is a *curve isometry* if for every piecewise smooth curve c in M, the image curve $f \circ c$ in $\mathbf{E}^N$ is rectifiable, and the length $L_M(c)$ of c in M equals the length $L_{\mathbf{E}^N}(f \circ c)$ of $f \circ c$ in $\mathbf{E}^N$.

Remark. Curve isometry is obviously a generalization of Riemannian isometry (i.e. f_* an isometry). However curve isometries are not necessarily differentiable (crease a piece of paper to obtain a counterexample: a non-differentiable curve-isometric immersion of a portion of $\mathbf{E}^2$ into $\mathbf{E}^3$).

Now let (r, s) and (u_1, u_2, v) denote Cartesian coordinates in $\mathbf{E}^2$ and $\mathbf{E}^3$ respectively.

Lemma 1. *Let* $f\colon \mathbf{E}^2 \to \mathbf{E}^3$ *be a curve isometry mapping the entire s-axis in $\mathbf{E}^2$ onto the v-axis in $\mathbf{E}^3$. Then* $f(r, s) = (u_1(r), u_2(r), s)$, *i.e. the image* $f(\mathbf{E}^2)$ *is a cylinder* $g \times V$, *where g denotes the plane curve* $(u_1(r), u_2(r))$ *and V denotes the v-axis.*

Remark. The differentiability of f is clearly dependent only on the differentiability of the curve g. If, for instance, g is piecewise differentiable, then the nondifferentiable points of f consist of lines parallel to the v-axis (creases!). Intuitively, if such a crease were not parallel to the s-axis in $\mathbf{E}^2$ it would intersect that axis, contradicting an assumption in the lemma.

The proof is based on the following considerations. If an arbitrary Riemannian manifold M is isometrically immersed in a Euclidean space $\mathbf{E}^n$ by a map f, then for any two points p, q in M, we have

$$d_M(p, q) \ge \|f(p) - f(q)\|,$$

where d_M and $\| \ \|$ denote, respectively, intrinsic Riemannian distance in M and Euclidean distance in E^n. This inequality derives, of course, from the definition of d_M, and so persists if f is only curve isometric in the sense of Lemma 1:

$$d_M(p, q) = \inf_c \{L_M(c) \,|\, c \text{ a curve in } M \text{ joining } p \text{ to } q\} = \inf_c \{L_{\mathbf{E}^N}(f \circ c)\}$$

$$\ge \inf_c \{L_{\mathbf{E}^N}(c) \,|\, c \text{ a curve in } \mathbf{E}^N \text{ joining } p \text{ to } q\}$$

$$= \|f(p) - f(q)\|.$$

We may now state the following principle.

The tetherball principle. Let $f\colon M \to \mathbf{E}^n$ be a curve isometry of a Riemannian manifold M into $\mathbf{E}^n$. Then if $f(p)$ is known for some fixed p in M, $f(q)$ must lie somewhere in the closed Euclidean ball of radius

$d_M(p, q)$ with center $f(p)$, where q is an arbitrary point in M. (Note that a tetherball, wherever it is located in space, must lie somewhere in a Euclidean ball of radius equal to the length of the tether rope with the top of the pole as center. Hence we refer to this principle as the tetherball principle.)

Proof of Lemma 1. Consider an arbitrary point q in E^2 with coordinates (r, s). We first show that $v(q) = s$. To do so, consider the point $p = (0, s + h)$. The distance from p to q in E^2 is exactly $(h^2 + r^2)^{\frac{1}{2}}$. Hence $f(q)$ must lie in the closed E^3-ball B_h of radius $(h^2 + r^2)^{\frac{1}{2}}$ and center $f(p) = (0, 0, s + h)$ by the tetherball principle. Similarly, using $p' = (0, s - h)$, we find $f(p)$ lies somewhere in the closed E^3-ball B_{-h} of radius $(h^2 + r^2)^{\frac{1}{2}}$ and center $(0, 0, s - h)$. It follows immediately that $(s + h) - (r^2 + h^2)^{\frac{1}{2}} \leq v(q) \leq (s - h) + (r^2 + h^2)^{\frac{1}{2}}$, and these inequalities hold for all h. As $h \rightarrow \infty$, we find $s \leq v(q) \leq s$ as required. Geometrically, we have $f(q) \in B_h \cap B_{-h}$, a lens-shaped region. As $h \rightarrow \infty$ these regions flatten out and tend in the limit to a disc of radius r lying in the plane $v = s$ in E^3. Hence, again $v(q) = s$.

Now let $q' = (r, s + h)$. We know $\|f(q') - f(q)\| \leq h$, but $v(q') - v(q) = h$; it follows that $u_i(q') = u_i(q)$, or the u_i are functions of r alone.

Lemma 1, as indicated by Hartman, can be extended in several ways. For one thing, the proof works equally well for E^p immersed in E^q, $q \geq p$. We will also need the following generalization.

Lemma 2. *Suppose* $M = M_1 \times E^1$ *is the Riemannian product of a connected Riemannian manifold* M_1 *and* E^1, *and suppose* $f: M \rightarrow E^N$ *is an isometric immersion mapping a geodesic* g *of form* $(p_1 \times E^1)$ *onto a straight line in* E^N. *Then* f *is cylindrical, i.e. there exists a decomposition* $E^N = E^{N-1} \times E^1$ *such that* $f = f_1 \times I$ *where* $f_1: M_1 \rightarrow E^{N-1}$ *is an isometric immersion and* $I: E^1 \rightarrow E^1$ *is the identity immersion (equivalently,* $f(p_1, p_2) = (f_1(p_1), p_2)$).

Proof. Join p_1 to a second point q_1 in M_1 by an arc g_1 in M_1. Now consider the two-dimensional submanifold $g_1 \times g$ of M. The metric on $g_1 \times g$ induced from M must be the product metric, hence it must be the flat metric. It follows that we can apply an obvious extension of Lemma 1 to $g_1 \times g$ immersed in E^N, to obtain the result needed.

Lemma 1 and its generalizations can be used in combination with other results to obtain further cylindricity theorems. For instance, together with results of Chern-Kuiper [3] on index of relative nullity, the well-known Hartman-Nirenberg theorem on the cylindricity of Euclidean hypersurfaces in Euclidean space is immediately obtained (in fact, $(M - K)$-cylindricity is obtained for E^M immersed in E^{M+K}, $K < M$). Lemma 1 makes it clear that differentiability is needed only for the local part of the theorem. (Compare [5], [7], [9], and [10] for other approaches to this theorem.)

Again following Hartman [5], and using the recent generalization of the Toponogov splitting theorem to the case of nonnegative Ricci curvature (due

to Cheeger-Gromoll [2]) we can state the following cylindricity theorem (see also Lichnerowicz [6] for a further generalization of the Cheeger-Gromoll result).

Theorem 1. *Let $f: M \to E^N$ be an isometric immersion of the complete Riemannian manifold M in the Euclidean space E^N. Assume further that M has nonnegative Ricci curvature, and that $\nu > 0$ everywhere, where ν denotes the index of relative nullity. Then $M = M_1 \times E^n$ and $f = f_1 \times I$ where f_1: $M_1 \to E^{N-n}$ and $I: E^n \to E^n$ is the identity immersion, and n is the minimal value of ν.*

Proof. By well-known results on index of relative nullity (see, e.g., [1]), there exist n linearly independent complete goedesics through some point in M which are mapped onto straight lines in E^N. By the Cheeger-Gromoll-Toponogov result, $M = M_1 \times E^n$. The cylindricity of the immersion now follows by Lemma 2.

Remark. Professor Hartman has pointed out in a private communication that the tetherball principle is equally valid if $f: M \to E^N$ is a curve *contraction* instead of an isometry (i.e. f *decreases* length of curves).

3. **Cylindricity of Riemannian product immersions.** In this section we use Hartman's lemma to give a simple proof of an unpublished cylindricity theorem of S. B. Alexander, which complements results of J. D. Moore [8] and Alexander-Maltz [1]. Here f nonflat means that f has a nonzero sectional curvature.

Theorem 2. *Let $f: M_1 \times \ldots \times M_k \times E^M \to E^N$ be an isometric immersion, each M_i a connected nonflat Riemannian manifold. Then the codimension C of the immersion is $\geq k$; if $C = k$, then f is cylindrical on the Euclidean factor E^M.*

Proof. We use the following inequality of [1]: $0 \leq \mu(p) - \nu(p) \leq C - i(p)$, where μ and ν denote index of nullity and relative nullity, respectively, and where $i(p)$ denotes the number of factors M_i with a nonzero sectional curvature at p_i, where $p = (p_i, \ldots, p_k, p_{k+1})$.

So by assumption there exists some point p at which $i(p) = k$, and $0 \leq \mu(p) - \nu(p) \leq C - k$. Hence $C \geq k$.

If $C = k$, then $\mu(p) = \nu(p) \geq M$. In particular, every geodesic of form $(p_1, \ldots, p_k, g)$ where g is a line in E^M is a nullity, and hence a relative nullity geodesic. As such it is mapped onto a straight line in E^N and the theorem follows immediately by Lemma 2, applied M times.

Remark. When the codimension equals k in Theorem 2, then not only is f cylindrical, but it actually decomposes into a product of immersions of the individual factors M_i and E^M, provided the M_i are compact [8] or satisfy Condition (∗) of [1].

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