SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS IN FUNCTION FIELDS OF ONE VARIABLE

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Abstract. Formal power series techniques are used to investigate the algebraic relationships between a function satisfying a linear differential equation and its derivatives. We are able to derive some conclusions, among them that an elliptic function satisfies no linear differential equation over a liouvillian extension of the complex numbers.

In [3], Rosenlicht noticed that if an element $y$ belonged to a liouvillian extension of a differential field, then the zeroes and poles of it and its derivatives must satisfy certain relations. His main tool was

Theorem. Let $K$ be a field of characteristic zero, $k$ a subfield of $K$, $P$ a real discrete $k$-place of $K$ whose residue field is algebraic over $k$, $D$ a derivation of $K$ that is continuous in the topology of $P$ and that maps $k$ into itself. Let $x, y$ be nonzero elements of $K$ such that each $x(P), y(P)$ is either $0$ or $\infty$. Then:

1. If $\text{ord}_P(Dx/x) \geq 0$, then $\text{ord}_P(Dy/y) \geq 0$. Here $D$ induces a derivation on the residue field of $P$. Denoting this residue field derivation by the same symbol $D$, for any $z$ in $K$ such that $\text{ord}_Pz \geq 0$, we have $(Dz)(P) = D(z(P))$.

2. If $\text{ord}_P(Dx/x) < 0$, then $\text{ord}_P(Dx/x) = \text{ord}_P(Dy/y)$ and, therefore, $\text{ord}_P(y/x) = \text{ord}_P(Dy/Dx)$. In addition, $(y/x)(P) = (Dy/Dx)(P)$.

Using this fact, he was able to show that certain differential equations have no liouvillian solutions. In this paper, we will show that the poles and zeroes of a solution of a linear differential equation and its derivatives must satisfy certain relations. With this we are able to mimic Rosenlicht’s results and show that solutions of a large class of differential equations satisfy no linear differential equation (Corollaries 1 and 2). We will also prove a strengthened version of results of C. L. Siegel [5, p. 60] and L. Goldman [1, Corollary 3] and give an easy proof of a structure theorem of L. Goldman [1, Corollary 4].

The main tool of this paper is

Lemma. Let $k \subseteq K$ be differential fields of characteristic 0. Let $w \in K$ satisfy the linear differential equation

$$w^{(n)} - A_{n-1}w^{(n-1)} - \cdots - A_0w = B$$

with the $A_i, B$ in $k$. Let $P$ be a discrete $k$-place of $k\langle w\rangle$ such that the derivation $'$ is continuous in the topology of this place. Then $\text{ord}_Pw < 0$ implies that $\text{ord}_P(w'/w) \geq 0$.

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PROOF. Assume not; then \( \text{ord}_P(w'/w) < 0 \) and so \( \text{ord}_P w' < 0 \). Case 2 of the theorem now applies. We can conclude that \( \text{ord}_P(w''/w') = \text{ord}_P(w/w) < 0 \) and \( \text{ord}_P w'' < 0 \). Similarly \( \text{ord}_P(w^{(k)}/w^{(k-1)}) < 0 \) and \( \text{ord}_P w^{(k)} < 0 \), and, in particular, \( \text{ord}_P(w^{(n)}/w^{(n-1)}) < 0 \). Using our linear equation, we have

\[
w^{(n)}/w^{(n-1)} = B/w^{(n-1)} + A_{n-1} + A_{n-2} w^{(n-2)}/w^{(n-1)} + \cdots + A_0 w/w^{(n-1)}.
\]

I claim that the right-hand side of this equation has order 0, which would give us a contradiction, and thus prove the lemma. First note that since

\[
\text{ord}_P(w^{(n-2)}/w^{(n-1)}) > 0 \quad \text{and} \quad \text{ord}_P(w^{(n-3)}/w^{(n-2)}) > 0,
\]

we have \( \text{ord}_P(w^{(n-3)}/w^{(n-1)}) > 0 \). Continuing in this way, we see that \( \text{ord}_P(w^{(n-1-i)}/w^{(n-1)}) > 0 \) for \( 2 \leq i \leq n-1 \). Also since \( \text{ord}_P w^{(n-1)} < 0 \), \( \text{ord}_P(B/w^{(n-1)}) > 0 \). Therefore, the order of the right-hand side of the above equation is 0.

COROLLARY 1. Let \( k \subseteq K \) be differential fields of characteristic 0 and \( y \in K \). Let \( f \) be a polynomial in several variables over \( k \) of total degree less than \( n \), some positive integer, and \( y^n = f(y, y', y'', \ldots) \). Assume further that the transcendence degree of \( k\langle y \rangle \) over \( k \) is 1. Then \( y \) satisfies no linear differential equation with coefficients in \( k \).

PROOF. Note that the transcendence degree assumption allows us to assume that the derivation is continuous in the topology of every \( k \)-plane [4, Lemma 1]. Assume that \( y \) did satisfy such an equation. By the lemma, we would then have \( \text{ord}_P(y'/y) \geq 0 \), where \( P \) is a pole of \( y \). This, in turn, implies that

\[
\text{ord}_P f(y, y', y'', \ldots) \geq (n-1)\text{ord}_P y > n(\text{ord}_P y)) = \text{ord}_P y^n,
\]

which is a contradiction.

COROLLARY 2. An elliptic function satisfies no linear differential equation with coefficients in a liouvillian extension of the complex numbers.

PROOF. Let \( k \) be a liouvillian extension of the complex numbers and \( y \) an elliptic function. Since \( y \) satisfies the differential equation \( (y')^2 = y^3 + ay + b \), for some \( a, b \in \mathbb{C}, a^3/27 + b^2/4 \neq 0 \), we could apply Corollary 1, once we know that the transcendence degree of \( k\langle y \rangle \) over \( k \) is 1. By looking at the above differential equation, we know it is at most 1. If it were less, then \( y \) would lie in a liouvillian extension of the complex numbers, contradicting the results on p. 372 of [3].

A homogeneous linear differential polynomial \( L(W) \), with coefficients in \( k \), is said to be linearly reducible over \( k \) if there exist homogeneous linear differential polynomials \( M(W), N(W) \), each of positive order, with coefficients in \( k \), such that \( L(W) = M(N(W)) \). If \( L(W) \) is not linearly reducible over \( k \), it is said to be irreducible over \( k \). We will need the following fact relating the reducibility of \( L(W) \) to the behavior of its solutions under isomorphisms. Let \( U \) be a universal extension of \( k \) with constant field \( C [2, p. 133], \) and \( x \) a nonzero element of \( U \) such that \( L(x) = 0 \). Let \( S \) be the set of differential \( k \)-isomorphisms of \( k(x) \) into \( U \). \( r \) the dimension of \( T \), the \( C \)-span of
\{\sigma x | \sigma \in S\} \) over \( C \), and \( n \) the order of \( L(W) \). I claim that there exist homogeneous linear differential polynomials \( L_{n-r}(W) \) and \( L_r(W) \), of order \( n - r \) and \( r \), with coefficients in \( k \), such that \( L(W) = L_{n-r}(L_r(W)) \).

To see this, we can assume that \( r \) is less than \( n \), and let \( \sigma_1 x, \sigma_2 x, \ldots, \sigma_r x \) be a \( C \)-basis of \( T \) and \( L_r(W) = \text{Wr}(W, \sigma_1 x, \ldots, \sigma_r x)/\text{Wr}(\sigma_1 x, \ldots, \sigma_r x) \), where \( \text{Wr}(\sigma_1, \ldots, \sigma_r) \) is the Wronskian determinant. Any isomorphism of \( k(\sigma_1, \ldots, \sigma_r) \) into \( U \) sends each \( \sigma_j x \) into \( T \) and so leaves the coefficients of \( L_r(W) \) fixed. By the corollary on p. 388 of [2], the coefficients of \( L_r(W) \) must be in \( k \). Let \( v_1 = \sigma_1 x, v_2 = \sigma_2 x, \ldots, v_r = \sigma_r x, v_{r+1}, \ldots, v_n \) be a fundamental system of solutions of \( L(W) \) in \( U \). Every differential \( k \)-isomorphism of \( k(\sigma_1, \ldots, \sigma_r) \) into \( U \) sends each \( v_j \) into the \( C \)-span of \( L(v_{r+1}), \ldots, L(v_n) \) and so leaves the coefficients of

\[
L_{n-r}(W) = \text{Wr}(W, L(v_{r+1}), \ldots, L(v_n))/\text{Wr}(L(v_{r+1}), \ldots, L(v_n))
\]

fixed. Therefore, \( L_{n-r}(W) \) also has its coefficients in \( k \). Since the coefficient of \( W^{(n)} \) in both \( L(W) \) and \( L_{n-r}(L_r(W)) \) is 1, \( L(W) - L_{n-r}(L_r(W)) \) is a homogeneous linear differential polynomial of order less than \( n \), with \( n \) linearly independent solutions. Therefore \( L(W) = L_{n-r}(L_r(W)) \). In particular, if \( L(W) \) is irreducible it has a fundamental set of solutions of the form \( x, \sigma_1 x, \ldots, \sigma_{n-1} x \), where \( x \) is any nonzero solution and the \( \sigma_i \)'s are differential \( k \)-isomorphisms of \( k(x) \) into \( U \).

**Corollary 3.** Let \( k \subset K \) be differential fields of characteristic 0 and \( w \in K \) which satisfies the linear differential equation \( L(W) = B \), where \( L(W) = W^{(n)} - A_{n-1} W^{(n-1)} - \cdots - A_0 W \) and the \( A_i \) and \( B \) are in \( K \). If the transcendence degree of \( k \langle w \rangle \) over \( k \) equals 1, then the homogeneous equation \( L(W) = 0 \) has a solution \( u \) such that \( u'/u \) is algebraic over \( k \). If \( L(W) \) is irreducible over \( k \), then \( L(W) = 0 \) has a fundamental set of solutions \( u_1, \ldots, u_n \) such that each \( u_i'/u_i \) is algebraic over \( k \).

**Proof.** The second assertion follows from the first and the remark at the end of the preceding paragraph.

To prove the first assertion, let \( P \) be a pole of \( w \). By the lemma, we have \( \text{ord}_P(w/w) \geq 0 \). Using case 1 of the Theorem, and observing that \( w' = w'w/w, w'' = ((w'/w)' + (w'/w)w)w, w''' = ((w'/w)'' + 3(w'/w)'w + (w'/w)^3)w, \ldots, w^{(n)} = ((w'/w)^{(n-1)} + n(w'/w)^{(n-2)}w + \cdots + (w'/w)^n)w, \) we see that \( (w'/w)(P) \) is an algebraic solution of the equation

\[
V^{(n-1)} + nV^{(n-2)} + \cdots + V^n - A_1(V^{(n-2)} + \cdots + V^{n-1}) - \cdots - A_n = (B/w)(P) = 0.
\]

We can now find a \( u \) in some differential extension field of \( k((w'/w)(P)) \) such that \( u'/u = (w'/w)(P) \). This \( u \) will then satisfy the homogeneous linear differential equation \( L(W) = 0 \).
COROLLARY 4. Let $k \subseteq K$ be differential fields of characteristic 0 and $z \in K$ a solution of a linear differential equation with coefficients in $k$. Assume that the transcendence degree of $k\langle z \rangle$ over $k$ is less than or equal to 1. Letting $\overline{k}$ be the algebraic closure of $k$, we can then find a $v$ in $\overline{k}\langle z \rangle$ such that $z$ is algebraic over $k\langle v \rangle$ and $v$ satisfies a linear differential equation of order 1 over $\overline{k}$.

PROOF. If the transcendence degree of $k\langle z \rangle$ over $k$ is zero, we are done. Assume $z$ is transcendental over $k$. Let $v$ be an element of $\overline{k}\langle z \rangle$, transcendental over $k$, which satisfies a linear differential equation over $\overline{k}$ of least order $r$, which we may assume is bigger than 1. Let $L(V) = B$ be a linear differential equation of order $r$ that $v$ satisfies. By Corollary 3 we know that $L(V) = 0$ has a solution $u$ such that $u'/u$ is in $\overline{k}$. Letting $S$ be the set of differential $\overline{k}$-isomorphisms of $\overline{k}\langle u \rangle$ into the universal domain $U$, the dimension of the $C$-span of $\{u\sigma|\sigma \in S\}$ is 1. Using the paragraph preceding Corollary 3, we can conclude that $L(V) = L_{r-1}(L_1(V))$, where $L_{r-1}(V)$ and $L_1(V)$ are homogeneous linear differential polynomials of order $r-1$ and 1, with coefficients in $\overline{k}$. $L_1(v)$ is in $\overline{k}\langle z \rangle$ and satisfies $L_{r-1}(V) = B$, a linear differential equation of order less than $r$. Therefore, $L_1(v)$ must be in $\overline{k}$ and $v$ satisfies a linear differential equation of order 1 over $\overline{k}$, a contradiction. Therefore, $r = 1$.

Both Corollary 4 and a weaker form of Corollary 3 were proven by L. Goldman [1] using the theory of differential polynomials and, in particular, the leading coefficient theorem of Ritt, which we have avoided.

REFERENCES


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