ON ARNOLD'S FORMULA FOR THE
DIMENSION OF A POLYNOMIAL RING

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Abstract. If $R$ is a commutative integral domain with quotient field $K$ and $x_1, \ldots, x_n$ are indeterminates, then there exist $\theta_1, \ldots, \theta_n$ in $K$ such that $\dim R[x_1, \ldots, x_n] = n + \dim R[\theta_1, \ldots, \theta_n].$

If $R$ is a commutative ring, the Krull dimension of $R$ is the maximum of the lengths of all chains of prime ideals in $R$. If $R = \mathbb{C}[V]$ is the coordinate ring of an affine variety $V$ over the complex numbers, then increasing chains of primes in $R$ correspond to decreasing chains of irreducible subvarieties. In this “geometric case” the Krull dimension corresponds to our intuitive notion of (complex) topological dimension. Moreover, since $R[X]$ corresponds to $V \times \mathbb{C}$ (the product of $V$ and an affine line), intuition would lead us to suspect

$$(*) \quad \dim R[X] = \dim R + 1.$$

In [7], W. Krull established $(*)$ for any noetherian ring. Seidenberg [9], [10] investigated the validity of $(*)$ for arbitrary commutative rings and observed that it does not hold in general. He observes that one always has

$$\dim R + 1 \leq \dim R[X] \leq 2 \dim R + 1,$$

and he provides examples to show that within these bounds anything can happen.

Jaffard [6] made an extensive study of the dimension theory in polynomial rings. He introduced the notion of valuative dimension of a domain $R$. This is just the maximum of the ranks of the valuation overrings of $R$. Jaffard showed that when $(*)$ fails, the valuative dimension of $R$ must exceed the dimension of $R$. In addition, he studied the asymptotic behavior of the function $f(n) = \dim R[x_1, \ldots, x_n]$ and showed that if $R$ is a domain of finite valuative dimension, then for all suitably large $n$ one has $f(n + 1) = f(n) + 1$.

In [4] Gilmer and Bastida call the sequence $\{f(i)\}_{i=0}^{\infty}$ the dimension sequence of the ring $R$, and they investigate which sequences are dimension sequences of a certain class of rings. In [2] Arnold and Gilmer determine all sequences which are the dimension sequence of a commutative ring.

Both [2] and [4] depend upon a result of Arnold [1, Theorem 5, p. 323] which we refer to as Arnold's formula. We state the result as follows:

Received by the editors September 12, 1974 and, in revised form, November 21, 1974.


Key words and phrases. Krull dimension, polynomial ring.

1 Supported by the National Science Foundation.

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If \( R \) is a commutative integral domain with quotient field \( K \) and \( X_1, \ldots, X_n \) are indeterminants over \( R \), then there exist \( \theta_1, \ldots, \theta_n \) in \( K \) such that

\[
(**) \quad \dim R[X_1, \ldots, X_n] = n + \dim R[\theta_1, \ldots, \theta_n].
\]

One always has that in (**) the left-hand side is greater than or equal to the right-hand side. Thus the interesting fact is that the maximum possible dimension of the rings of the form \( R[\theta_1, \ldots, \theta_n] \) can always be realized. The proof of the formula in [1] is, however, incorrect and we know of no correct proof in the literature.\(^2\)\(^3\) Our purpose here is to provide an elementary proof.

In what follows, all rings are assumed to be commutative and to possess an identity. When we write "dim \( R \)" we are referring to the Krull dimension of the ring \( R \). By \( R[X_1, \ldots, X_n] \) we denote the ring of polynomials in the independent variables \( \{X_1, \ldots, X_n\} \) over the ring \( R \). Finally, whenever we use the symbol "<" it is meant to denote strict containment.

Our argument requires a few well-known facts which we list for the convenience of the reader.

(A) Let \( k \) be a field and \( R = k[x_1, \ldots, x_n] \) a finitely generated ring extension of \( k \). If \( P \) is a prime of \( R \), then \( \dim R = \text{rank } P + \text{trans deg}_k (R/P) \) [8 , p.46, (14.6)].

The following is observed by Seidenberg [9] and is a consequence of (A) applied to the ring of polynomials in \( n \) variables over a field.

(B) If \( \{P_i\}_{i=0}^l \) is a chain of primes in \( R[X_1, \ldots, X_n] \) all of which lie over the same prime of \( R \), then \( l < n \).

(C) If \( V^* \) is a valuation ring of rank \( n \) with quotient field \( L \), and if \( L \) is of transcendence degree \( d \) over the field \( K \), then \( \text{rank } (V^* 
 K) > n - d \) [3, p.440, Corollary 2].

A chain \( \mathfrak{D} = \{P_i\}_{i=0}^m \) of primes in a polynomial ring \( R[X_1, \ldots, X_n] \) is called a special chain if, for each \( P_i \in \mathfrak{D} \), the ideal \( (P_i \cap R)[X_1, \ldots, X_n] \) is a member of \( \mathfrak{D} \).

(D) Jaffard's special chain theorem. If \( Q \) is a prime ideal of \( R[X_1, \ldots, X_n] \) of finite rank \( r(Q) \), then \( r(Q) \) can be realized as the length of a special chain of primes in \( R[X_1, \ldots, X_n] \) with terminal element \( Q \). In particular, if \( R \) is finite dimensional then \( \dim R[X_1, \ldots, X_n] \) can be realized as the length of a special chain of primes of \( R[X_1, \ldots, X_n] \).

This is the statement of Jaffard's theorem in [5]. The authors give an argument there which we feel is much easier than Jaffard's proof [6].

Proof of Arnold's formula. Suppose there were a counterexample, say \( R[X_1, \ldots, X_n] \). Then we may assume that \( n \) is minimal and that for this fixed \( n \), \( R \) has minimal dimension. We must have that both \( n \) and \( \dim R \) are greater than zero: if \( \dim R = 0 \), \( R \) is a field and by (A) we could simply take \( \theta's = 0 \). If \( n = 0 \) we could again take all \( \theta's = 0 \).

\(^2\) I wish to thank Jon Johnson for bringing the error to my attention.

\(^3\) Bastida and Gilmer point out the error in [4]. However their discussion of the mistake is itself erroneous. In [4] they claim to remove the doubt about this formula by independently proving Theorem 3 of [1] (which has the incorrect proof). However their argument ultimately rests on Theorem 5 of [1] whose proof (in [1]) is based on Theorem 3 of [1].
Let \( s = \dim R[X_1, \ldots, X_n] \). By (D) there is a special chain \( 0 < P_1 < \cdots < P_t < \cdots < P_n \) in \( R[X_1, \ldots, X_n] \). Let \( t \) be minimal such that \( P_t \cap R \neq 0 \). Let \( P_t \cap R = q \). By (A), \( t - 1 \leq n \) and \( P_t = q[X_1, \ldots, X_n] \) by the minimality of \( t \). Since our chain has maximal length, rank \( q[X_1, \ldots, X_n] = t \). Let \( S = R \setminus q \) and localize \( R[X_1, \ldots, X_n] \) at \( S \) to get \( T = R_q[X_1, \ldots, X_n] \). Now for any local ring \( R_q \), it is an easy consequence of (D) that \( \dim R_q[X_1, \ldots, X_n] = n + \text{rank } q[X_1, \ldots, X_n] \) (or see [5]). Thus \( \dim T = n + t \).

Let \( T = R_q[X_1, \ldots, X_n]/P_{t-1}R_q[X_1, \ldots, X_n] \). Then \( \dim T = n + 1 \) and there exists a chain of primes

\[
0 < qT < Q_2 < \cdots < Q_{n+1} < T.
\]

Since \( P_{t-1} \cap R = 0 \) we may identify \( R \) with its image in \( T \) and assume that \( K \), the quotient field of \( R \), is contained in that of \( T \). Let \( V^* \) be a valuation overring or \( T \) which is centered on the chain \((***)\) [8, p.37, (11.9)]. Let \( V = V^* \cap K \).

**Claim.** rank \( V \geq t \).

**Proof of Claim.** We first compute the transcendence degree of \( T \) over \( R \). To do this we may localize \( T \) at \( s' = R \setminus 0 \). By the permutability of residue class ring and quotient ring formation

\[
\overline{T} = (R_q[X_1, \ldots, X_n]/P_{t-1}R_q[X_1, \ldots, X_n])_{s'} = K[X_1, \ldots, X_n]/P_{t-1}K[X_1, \ldots, X_n].
\]

Since rank \( P_{t-1} = \text{rank } P_{t-1}(K[X_1, \ldots, X_n]) = t - 1 \), we have \( \dim \overline{T} = [n - (t - 1)] \) (by (A)). Thus by (A) the transcendence degree of \( T \) over \( R \) is \( [n - (t - 1)] \). Therefore we have by (C) that \( \text{rank } (V') \geq (n + 1) - [n - (t - 1)] = t \). This establishes the Claim.

Since each prime of the chain \((***)\) contains \( q \), it follows that each prime of \( V \) contains \( q \). Thus each prime of \( V \) meets \( R_q \) at \( qR_q \). Let \( M_1 < \cdots < M_t \) be a chain in \( V \) such that \( M_j \cap R_q = qR_q \). For each \( j \) such that \( 1 \leq j \leq t - 1 \), choose \( \theta_j \in M_{j+1} \setminus M_j \).

Consider the canonical homomorphism \( V \to V/M_t \). Under \( \sigma \), \( R_q[\theta_1, \ldots, \theta_{t-1}] \) maps into \( V/M_t \). Denote the field \( R_q/qR_q \) by \( k \) and let \( \sigma(\theta_j) = \bar{\theta}_j \). Then under \( \sigma \) we have

\[
R/q[\bar{\theta}_1, \ldots, \bar{\theta}_{t-1}] \subset k[\bar{\theta}_1, \ldots, \bar{\theta}_{t-1}] \subset V/M_t.
\]

By our choice of the \( \theta_j \)'s, the primes \( M_j/M_t \) lie over distinct primes of \( k[\bar{\theta}_1, \ldots, \bar{\theta}_{j-1}] \) for \( j = 1, \ldots, t \). Thus \( \dim k[\bar{\theta}_1, \ldots, \bar{\theta}_{t-1}] \geq t - 1 \) and by (A) the elements \( \bar{\theta}_1, \ldots, \bar{\theta}_{t-1} \) are algebraically independent over \( k \).

Let \( I \) denote the kernel of the homomorphism

\[
R[X_1, \ldots, X_{t-1}] \to R[\theta_1, \ldots, \theta_{t-1}]
\]

given by \( X_j \to \theta_j \). If we follow \( \tau \) by \( \sigma \) we get a mapping

\[
R[X_1, \ldots, X_{t-1}] \xrightarrow{\sigma \circ \tau} (R/q)[\bar{\theta}_1, \ldots, \bar{\theta}_{t-1}] \cong (R/q)[X_1, \ldots, X_{t-1}]
\]

which clearly has kernel \( I \subset q[X_1, \ldots, X_{t-1}] \). Thus \( I \subset q[X_1, \ldots, X_{t-1}] \). Moreover, since \( I \cap R = 0 \), the containment is strict. Again using the fact that
at the multiplicative system $R \setminus 0$. This yields an exact sequence
\[ 0 \to I \to K[X_1, \ldots, X_{i-1}] \to K[X_1, \ldots, X_{i-1}] \to K \to 0. \]
Thus $I \cdot (K[X_1, \ldots, X_{i-1}])$ has rank $t - 1$ and it follows that $I$ has rank $t - 1$. Recalling that $t - 1 \leq n$, we have now shown that $I = q[X_1, \ldots, X_n]$ contains the prime $I[X_1, \ldots, X_n]$ which is the kernel of the homomorphism
\[ R[X_1, \ldots, X_n] \to R[\theta_1, \ldots, \theta_{i-1}][X_1, \ldots, X_n] \]
which takes $X_i$ to $\theta_i$ if $1 \leq i \leq t - 1$ and $X_j$ to $X_j$ if $j \geq t$. Since
\[ t - 1 = \text{rank } I \leq \text{rank } I[X_1, \ldots, X_n] < \text{rank } q[X_1, \ldots, X_n] = t, \]
we see that rank $I[X_1, \ldots, X_n] = t - 1$. We can now modify our original chain $0 < P_1 < \cdots < P_i < \cdots < P_n$ so that $P_{i-1} = I[X_1, \ldots, X_n]$. Hence we have the computation
\[ [s - (t - 1)] = \dim R[X_1, \ldots, X_n]/P_{i-1} = \dim R[X_1, \ldots, X_n]/I[X_1, \ldots, X_n] = \dim R[\theta_1, \ldots, \theta_{i-1}][X_1, \ldots, X_n]. \]
There are now two possibilities, each of which leads to a contradiction.

**Case 1.** $t > 1$. In this case, by the minimality of $n$ there exist $\gamma_1, \ldots, \gamma_n$ in $K$ such that
\[ s - (t - 1) = \dim R[\theta_1, \ldots, \theta_{i-1}, X_1, \ldots, X_n] = [n - (t - 1)] + \dim R[\theta_1, \ldots, \theta_{i-1}, \gamma_1, \ldots, \gamma_n]. \]
That is,
\[ \dim R[\theta_1, \ldots, \theta_{i-1}, \gamma_1, \ldots, \gamma_n] + n = s = \dim R[X_1, \ldots, X_n]. \]
This is a contradiction.

**Case 2.** $t = 1$. In this case $P_1 = q[X_1, \ldots, X_n]$ and we have
\[ s - 1 = R[X_1, \ldots, X_n]/q[X_1, \ldots, X_n] = R/q[X_1, \ldots, X_n]. \]
By the minimality of $\dim R$, there are $\theta_1, \ldots, \theta_n$ in the quotient field of $R/q$ such that
\[ \dim (R/q)[X_1, \ldots, X_n] = \dim (R/q)[\theta_1, \ldots, \theta_n] + n. \]
Let $\lambda : R \to R/q$ be the canonical homomorphism. Then $\lambda$ extends to a mapping of $R_q$ onto the quotient field of $R/q$. Choose $\theta_1, \ldots, \theta_n$ in $R_q$ such that $\lambda(\theta_i) = \theta_i$. Then we have an induced homomorphism
\[ R[\theta_1, \ldots, \theta_n] \xrightarrow{\lambda} (R/q)[\theta_1, \ldots, \theta_n]. \]
Since this homomorphism has a nontrivial kernel,
\[ \dim \mathcal{R} = \dim (R/q) \mathcal{X} + n \leq \dim \mathcal{R}[X_1, \ldots, X_n]. \]

Thus to complete our argument we need only show that \( \dim \mathcal{R}[X_1, \ldots, X_n] + n < \dim \mathcal{R}[X_1, \ldots, X_n] \). But, as mentioned earlier, this is true for any set of \( \theta \)'s. For consider the exact sequence

\[ 0 \to J \to \mathcal{R}[X_1, \ldots, X_n] \to \mathcal{R}[\theta_1, \ldots, \theta_n] \to 0 \]

given by the homomorphism \( X_j \to \theta_j \). We surely have \( \dim \mathcal{R}[X_1, \ldots, X_n] \geq \dim \mathcal{R}[\theta_1, \ldots, \theta_n] + \text{rank } J \). We need only show that rank \( J = n \). Since \( J \cap \mathcal{R} = 0 \), we may localize at the multiplicative system \( \mathcal{R}\setminus 0 = \mathcal{S} \) and compute the rank of \( J \). After localizing we have

\[ 0 \to J \to K[X_1, \ldots, X_n] \to K \to 0. \]

If we now apply (A) we compute rank \( J = n \).

REFERENCES


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