ON THE PLANE TERM RANK OF A THREE
DIMENSIONAL MATRIX

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Abstract. A lower bound is obtained for the plane term rank of a three
dimensional matrix in which each of the horizontal two dimensional matrices
has constant row and column sums.

1. Introduction. Let \( A = [a_{ijk}] \), \( 1 \leq i, j, k \leq n \), be a three dimensional
matrix of 0's and 1's of order \( n \). A plane (respectively, line) of \( A \) is defined to
be the two dimensional matrix (respectively, one dimensional matrix or vector)
which results when one (respectively, two) of \( i,j,k \) is held fixed. We distinguish
three types of planes: the row planes \( (i \) fixed), the column planes \( (j \) fixed) and
the horizontal planes \( (k \) fixed). Each line of \( A \) is a line (that is, row or column)
of two planes of \( A \). The plane term rank of \( A \), \( \rho(A) \), is \( r \) if \( r \) is the largest number
such that there exist \( r \) entries of \( A \) which equal 1 with no two of the entries in
the same plane of \( A \). The plane term rank of \( A \) is one of the two possible
generalizations of the term rank of a two dimensional matrix of 0's and 1's. It
is well known that the analogue for the plane term rank of König's theorem
concerning the equality of the term rank and covering number of a two
dimensional matrix is false. It is thus of some interest to obtain estimates for
the plane term rank.

In [1] we proved that if \( A \) is a three dimensional matrix of 0's and 1's of
order \( n \) with exactly \( c \) 1's in each line, then the plane term rank satisfies
\( \rho(A) \geq \min\{n/2, c\} \).

Koksma [2] has proved that a latin square of order \( n \) has a partial
transversal of length at least \( (2/3)n + 1/3 \). A latin square \( L \) of order \( n \) can be
regarded as a three dimensional matrix \( A(L) = [a_{ijk}] \) of 0's and 1's of order \( n \)
whereby \( a_{ijk} = 1 \) if and only if the \((i,j)\)-entry of \( L \) is \( k \). Such a matrix has
exactly one 1 in each line; conversely, every matrix of this type corresponds
to a latin square of order \( n \). A partial transversal of length \( r \) of the latin square
\( L \) corresponds to a set of \( r \) 1's of the matrix \( A(L) \) with no two of the 1's in the
same plane. Thus Koksma's result can be restated as follows: If \( A \) is a three
dimensional matrix of 0's and 1's of order \( n \) with exactly one 1 in each line,
then \( \rho(A) \geq (2/3)n + 1/3 \). In this note we obtain a bound for the plane term
rank of three dimensional matrices in which each of the horizontal planes has
constant row and column sums (which may vary with the horizontal plane).

2. Results. For a real number \( a \) we set \( p(a) = \lfloor a \rfloor \) if \( a \) is not an integer and
\( p(a) = a - 1 \) if \( a \) is an integer.

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Theorem 1. Let $A$ be a three dimensional matrix of 0's and 1's of order $n$ with the following properties:

1. For each $k = 1, \ldots, n$ the $k$th horizontal plane has exactly $c_k$ 1's in each row and column.

2. For an integer $c$ with $c_k \geq c$ $(k = 1, \ldots, n)$, each of the row planes of $A$ has no $n - 1 - p(c/(c + 1)n)$ by $2n - 2p(c/(c + 1)n)$ submatrix of all 0's.

Then $\rho(A) \geq (c/(c + 1)n)$.

Proof. Let $\rho(A) = t$. Without loss of generality, we may assume that $a_{kkk} = 1$ for $k = 1, \ldots, t$. For each $k = 1, \ldots, n$ we consider the $k$th horizontal plane $H_k$ to be partitioned into submatrices $H_{k,1}, H_{k,2}, H_{k,3}, H_{k,4}$ as indicated in the figure below. Since $\rho(A) = t$, $H_{k,4}$ is a matrix of all 0's for $k = t + 1, \ldots, n$.

Consider the horizontal plane $H_{t+1}$. Suppose no two 1's in $H_{t+1,3}$ were in the same column. Since there are $c_{t+1}(n - t)$ 1's in $H_{t+1,3}$, this would imply that $t \geq c_{t+1}(n - t) \geq c(n - t)$, so that $t \geq (c/(c + 1)n)$. Thus we may suppose that for some $j_0$ with $1 \leq j_0 \leq t$, column $j_0$ of $H_{t+1,3}$ contains a 1 in the $p$th row and $q$th row where $1 \leq p < q \leq n - t$.

Suppose there were a 1 in $H_{j_0,4}$. This 1 cannot be in both the $p$th row and $q$th row of $H_{j_0,4}$. Without loss of generality, we may assume it is in row $u \neq p$; let this be in the column $v$ of $H_{j_0,4}$. Then each of the positions $(1,1,1), \ldots, (j_0 - 1, j_0 - 1, j_0 - 1), (j_0 + 1, j_0 + 1, j_0 + 1), \ldots, (t,t,t), (t + p,j_0,t + 1), (t + u,t + v,j_0)$ of $A$ is occupied by a 1, and this implies that $\rho(A) > t + 1$. This is a contradiction, so that we conclude that $H_{j_0,4}$ is a matrix of 0's. Thus the $c_{j_0}(n - t)$ 1's in columns $t + 1, \ldots, n$ of $H_{j_0}$ occur in $H_{j_0,2}$. Since there are $c_{j_0}$ 1's in each row of $H_{j_0}$, this implies that $H_{j_0,2}$ has 1's in at least $n - t$ different rows.

In addition, row $j_0$ of $H_{j_0,2}, H_{t+2,2}, \ldots, H_{n,2}$ must consist of all 0's. For if there were a 1 in row $j_0$ of any of these, then this 1, the specified 1 in row $p$ of $H_{t+1,3}$ and the 1's in positions $(1,1,1), \ldots, (j_0 - 1, j_0 - 1, j_0 - 1), (j_0 + 1, j_0 + 1, j_0 + 1), \ldots, (t,t,t)$ would imply that $\rho(A) > t + 1$, a contradiction.

Let rows $i_1, \ldots, i_{n-t}$ of $H_{j_0,2}$ each contain at least one 1 where $1 \leq i_1 < \cdots < i_{n-t} < t$ and $i_k \neq j_0$ for $k = 1, \ldots, n - t$. Consider an $i_r$. If the $(j_0,i_r)$-entry of any of $H_{t+2,2}, \ldots, H_n$ were a 1, then a 1 in row $i_r$ of $H_{j_0,2}$ in the $(j_0,i_r)$-position of one of $H_{t+2,2}, \ldots, H_n$ the specified 1 in row $p$ of $H_{t+1,3}$ along with the 1's in the $t - 2$ positions of the set

\[
\{(1,1,1), \ldots,(t,t,t)\} \setminus \{(j_0,j_0,j_0),(i_r,i_r,i_r)\}
\]

lead to the conclusion again that $\rho(A) \geq t + 1$. Thus the $(j_0,i_r)$-entry of each of $H_{t+2,2}, \ldots, H_n$ is 0.
We conclude that the submatrix of the \( j_0 \)th row plane \( R_j \) of \( A \) which lies at the intersection of rows \( t + 2, \ldots, n \) and columns \( i_1, \ldots, i_{n-t}, t+1, \ldots, n \) consists of all 0's. Thus \( R_j \) has an \( n - t - 1 \) by \( 2(n - t) \) submatrix of all 0's.

Now suppose that \( t < \left( \frac{c}{(c + 1)^2} \right)n \), so that \( t \leq p(\frac{c}{(c + 1)^2})n \). Then

\[
n - t - 1 \geq n - 1 - p\left(\frac{c}{c + 1}\right)n \quad \text{and} \quad 2(n - t) \geq 2n - 2p\left(\frac{c}{c + 1}\right)n,
\]

and we have contradicted (2). We conclude that \( t \geq \left( \frac{c}{(c + 1)^2} \right)n \), and the theorem is proved.

Recall that an \( n \times n \) matrix of 0's and 1's is called fully indecomposable if it has no \( r \times s \) submatrix of 0's with \( r + s = n \) \((r, s \geq 1)\).

**THEOREM 2.** Let \( A \) be a three dimensional matrix of 0's and 1's of order \( n \) satisfying:

3. For each \( k = 1, \ldots, n \) the \( k \)th horizontal plane has \( c_k \) \((\geq 2)\) 1's in each row and column.

4. Each of the row planes of \( A \) is a fully indecomposable matrix. Then \( \rho(A) \geq (2/3)n \).

**PROOF.** By taking \( c = 2 \) in Theorem 1 it is enough to show that the hypotheses imply that each of the row planes of \( A \) has no \( n - 1 - p(2/3)n \) by \( 2n - 2p(2/3)n \) submatrix of all 0's. But

\[
n - 1 - p\left(\frac{2}{3}\right)n + 2n - 2p\left(\frac{2}{3}\right)n = 3n - 1 - 3p\left(\frac{2}{3}\right)n
\]

which equals \( n + 2, n + 1, \) or \( n \) according as whether \( n \) is congruent to 0, 1, or 2 modulo 3. Since each row plane of \( A \) is assumed to be fully indecomposable, the theorem follows from Theorem 1.

**REFERENCES**


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