A THEOREM ON NILPOTENT GROUPS
WITH RESTRICTED EMBEDDINGS

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Abstract. Suppose that $G$ is a nonabelian group with a unique proper normal subgroup of some given order. It is proved that $G$ is not contained as a normal subgroup within the Frattini subgroup of a finite $p$-group.

Burnside [1] has proved that nonabelian groups with cyclic center, or whose derived subgroup has index $p^2$ cannot occur as the derived subgroup of a $p$-group. Hobby [2] showed that the same classes of groups also cannot occur as the Frattini subgroup of a $p$-group. Nonabelian $p$-groups with cyclic center or whose derived subgroup has index $p^2$ are examples of groups which have a unique proper normal subgroup of some given order. In this note we prove that the theorems of Burnside and Hobby may be generalized to include this wider class of $p$-groups. In fact we prove the following more general theorem.$^1$

Theorem. Suppose $G$ is a nonabelian group with a unique proper normal subgroup of some given order. Then $G$ is not contained as a normal subgroup within the Frattini subgroup of a finite $p$-group.

Proof. Denote by $G_0$ the unique proper normal subgroup of $G$. Suppose first that $G_0$ is cyclic and that $G$ is a normal subgroup of a $p$-group $P$ within the Frattini subgroup $\Phi(P)$. By [3, Satz III 7.5] there exists an elementary abelian subgroup $R$ of $G$ which is normal in $P$ and has type $(p,p)$. Since $G_0$ is cyclic it is contained in $R$ and so has order $p$. As $G_0$ is the only normal subgroup of $G$ having order $p$, it follows that $Z(G)$ is cyclic. However, since $R$ is normal in $P$, the order of $P/C_p(R)$ divides the order of the group of automorphisms of $R$. Hence, $C_p(R)$ is either $P$ itself or a maximal subgroup of $P$.

In either case $G \leq \Phi(P) \leq C_p(R)$ and, hence, $R \leq Z(G)$ contrary to the fact that $Z(G)$ is cyclic. Thus the theorem holds if $G_0$ is cyclic.

To complete the proof we use induction on the order of $G_0$. Suppose $G_0$ is noncyclic and $G$ is a normal subgroup of a $p$-group $P$ within the Frattini subgroup $\Phi(P)$. Let $L$ be a normal subgroup of $P$ of order $p$ contained in $G_0$; then $G_0/L$ is a unique proper normal subgroup of $G/L$ of the given order $|G_0/L| < |G_0|$. Also $G/L \leq \Phi(P)/L = \Phi(P/L)$ and by induction $G/L$ is abelian and, hence, cyclic. Since $L \leq Z(G)$, $G$ is abelian, contrary to the hypothesis of the theorem. This completes the proof of the theorem.$^1$

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REFERENCES


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