A COEFFICIENT ESTIMATE FOR MULTIVALENT FUNCTIONS

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Abstract. By making use of extreme point theory we obtain bounds on the coefficients of a class of functions, multivalent in the unit disk, closely related to the bounds conjectured by Goodman.

I. Introduction. Let \( S(p,q) \), \( p \) and \( q \) integers, \( 1 < q < p \), be the class of functions \( f(z) \) analytic in \( \Delta = \{ z : |z| < 1 \} \) with power series expansion

\[
f(z) = \sum_{n=q+1}^{\infty} a_n z^n, \quad z \in \Delta,
\]

and for which there exists a \( \rho = \rho(f) \) such that

\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0
\]

and

\[
\int_0^{2\pi} \text{Re} \left( \frac{zf'(z)}{f(z)} \right) d\theta = 2p\pi \quad \text{for } z = re^{i\theta}, \rho < r < 1.
\]

Functions in \( S(p,q) \) are \( p \)-valent and are referred to as multivalent starlike functions [2]. We let \( S_1(p,q) \) be the subclass of \( S(p,q) \) of functions which are analytic on \( |z| = 1 \) and satisfy (1.1) and (1.2) on \( |z| = 1 \).

We define the class \( K(p,q) \) [5] to be the class of functions \( f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n, \ z \in \Delta \), for which there exists \( g(z) \) in some \( S(p,i) \), an \( \alpha \), \( 0 < \alpha < 2\pi \), and a \( \rho \), \( 0 < \rho < 1 \), such that

\[
\text{Re} \left( e^{i\alpha} \frac{zf'(z)}{g(z)} \right) > 0 \quad \text{for } \rho < |z| < 1.
\]

Functions in \( K(p,q) \) are called multivalent close-to-convex functions. We let \( K_1(p,q) \) be the subclass of functions \( f(z) \) in \( K(p,q) \) which are analytic on \( |z| = 1 \) and for which there exists \( g(z) \) in some \( S_1(p,i) \) and an \( \alpha \), \( 0 < \alpha < 2\pi \), such that (1.3) holds on \( |z| = 1 \). It is known [5] that if \( f(z) \) is in \( K(p,q) \), \( zf'(z) \) has exactly \( p \) zeros in \( \Delta \). We thus divide \( K(p,q) \) into subclasses according to the location of the nonzero zeros of \( zf'(z) \). Let \( \alpha_i, \ i = 1,2, \ldots, p-q \), be arbitrary complex numbers satisfying \( 0 < |\alpha_i| < 1, \ i = 1,2, \ldots, p-q \), and define \( K(p,q,\alpha_1, \alpha_2, \ldots, \alpha_{p-q}) \) to be the class of functions \( f(z) \) such that \( f(z) \) is in \( K(p,q) \) and \( zf'(z) = 0 \) for \( z = \alpha_i, \ i = 1,2, \ldots, p-q \). Furthermore we let \( K_1(p,q,\alpha_1, \ldots, \alpha_{p-q}) \) be the subclass of functions in...
A class related to $K(p,q,\alpha_1,\ldots,\alpha_{p-q})$ is the class $\hat{K}(p,q,\alpha_1,\ldots,\alpha_{p-q})$ defined by the following: $f(z)$ is in $\hat{K}(p,q,\alpha_1,\ldots,\alpha_{p-q})$ if $f(z)$ is analytic in $\Delta$, $f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n$ for $z$ in $\Delta$ and $zf'(z)$ has the representation

$$zf'(z) = \frac{\prod_{i=1}^{p-q}(z - \alpha_i)(1 - \bar{\alpha}_i z)}{z^{p-q}\prod_{i=1}^{p-q}(-\alpha_i)} p(z)h(z)$$

where $h(z)$ is in $S(p,p)$ and $p(z)$ satisfies $p(0) = q$ and there exists $\delta > 0$ such that $\Re e^{i\delta} p(z) > 0$ for $z$ in $\Delta$.

It has been conjectured by Goodman [1] that for a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, which is analytic and $p$-valent in $\Delta$, the coefficients satisfy the inequalities

$$2k(n+p)! \leq \sum_{k=1}^{p} (p+k)!(p-k)!(n-p-1)!(n^2 - k^2) |a_k|$$

for $n \geq p + 1$. These inequalities are known to be true for $K(p,p)$ and $K(p,p-1)$ [6] but not for $K(p,q)$ if $q < p - 1$. The situation is similar for the classes $S(p,q)$ except that the inequalities are known to be true for functions in $S(p,q)$, $1 \leq q < p$, whose power series have real coefficients [3]. In §III of this paper we show that by making use of linear methods, inequalities closely related to those conjectured by Goodman [1] can be proven for the class $K(p,q,\alpha_1,\ldots,\alpha_{p-q})$.

II. Classes of functions related to $K(p,q)$.

Theorem 1. $K(p,q,\alpha_1,\ldots,\alpha_{p-q}) \subset \hat{K}(p,q,\alpha_1,\ldots,\alpha_{p-q})$.

Proof. Suppose first that $f(z)$ is in $K(p,q,\alpha_1,\ldots,\alpha_{p-q})$; then [5] there exists $h(z)$ in $S_1(p,p)$ and a $\beta$ such that $\Re [e^{i\beta} z f'(z)/h(z)] > 0$ on $|z| = 1$. As in [5] we easily see that

$$g(z) = \frac{\prod_{i=1}^{p-q}(z - \alpha_i)(1 - \bar{\alpha}_i z)}{z^{p-q}\prod_{i=1}^{p-q}(-\alpha_i)} h(z)$$

is in $S_1(p,q)$. Let $\delta = \beta - \arg \prod_{i=1}^{p-q}(-\alpha_i)$; then $\Re e^{i\delta} [z f'(z)/g(z)] > 0$ for $|z| = 1$. But since $z f'(z)/g(z)$ is analytic for $|z| < 1$ we have $\Re (e^{i\delta} z f'(z)/g(z)) > 0$ for $|z| < 1$. Letting $p(z) = z f'(z)/g(z)$ we obtain (1.1) for $z f'(z)$.

Next, if $f(z)$ is analytic only for $|z| < 1$, there exists a $\rho$, $0 < \rho < 1$, such that $g_r(z) = r^{-\delta} f(rz)$ is in $K_0(p,q)$ for $\rho < r < 1$. Since $g_r(z) = 0$ for $z = \alpha_i/r$, $i = 1,2,\ldots, p - q$, we have

$$g_r(z) = \frac{\prod_{i=1}^{p-q}(z - \alpha_i/r)(1 - \bar{\alpha}_i z/r)}{z^{p-q}\prod_{i=1}^{p-q}(-\alpha_i/r)} p_r(z)h_r(z)$$

These inequalities are known to be true for $K(p,p)$ and $K(p,p-1)$ [6] but not for $K(p,q)$ if $q < p - 1$. The situation is similar for the classes $S(p,q)$ except that the inequalities are known to be true for functions in $S(p,q)$, $1 \leq q < p$, whose power series have real coefficients [3]. In §III of this paper we show that by making use of linear methods, inequalities closely related to those conjectured by Goodman [1] can be proven for the class $K(p,q,\alpha_1,\ldots,\alpha_{p-q})$.

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is in $S_1(p,q)$. Let $\delta = \beta - \arg \prod_{i=1}^{p-q}(-\alpha_i)$; then $\Re e^{i\delta} [z f'(z)/g(z)] > 0$ for $|z| = 1$. But since $z f'(z)/g(z)$ is analytic for $|z| < 1$ we have $\Re (e^{i\delta} z f'(z)/g(z)) > 0$ for $|z| < 1$. Letting $p(z) = z f'(z)/g(z)$ we obtain (1.1) for $z f'(z)$.

Next, if $f(z)$ is analytic only for $|z| < 1$, there exists a $\rho$, $0 < \rho < 1$, such that $g_r(z) = r^{-\delta} f(rz)$ is in $K_0(p,q)$ for $\rho < r < 1$. Since $g_r(z) = 0$ for $z = \alpha_i/r$, $i = 1,2,\ldots, p - q$, we have

$$g_r(z) = \frac{\prod_{i=1}^{p-q}(z - \alpha_i/r)(1 - \bar{\alpha}_i z/r)}{z^{p-q}\prod_{i=1}^{p-q}(-\alpha_i/r)} p_r(z)h_r(z)$$


where $h_\ell(z)$ is in $S_\ell(p,q)$, $p(0) = q$ and there exists $\delta_\ell$ such that $\text{Re}(e^{i\delta\ell h_\ell(z)}) > 0$. Using the fact that the families of functions $e^{i\delta\ell h_\ell(z)}$ and $h_\ell(z)$ belong to normal and compact families, we easily obtain (1.1) from (1.5) upon passing to the limit.

We define the class $\hat{C}(p,q,\alpha_1, \ldots, \alpha_{p-q})$ to be the class of functions $f(z)$, analytic in $\Delta$, such that $zf'(z)$ has the representation

\begin{equation}
zf'(z) = \frac{\prod_{i=1}^{p-q}(z - \alpha_i)(1 - \overline{\alpha_i} z)}{z^{p-q} \prod_{i=1}^{p-q}(-\alpha_i)} \cdot p(z)z^p \prod_{s=1}^{2p}(1 - e^{-i\theta_s}z)^{-1}
\end{equation}

where $\alpha_i, 0 < |\alpha_i| < 1$, are fixed for $i = 1,2, \ldots, p-q$; $p(0) = q$ and there exists $\delta$ so that $\text{Re}(e^{i\delta p(z)}) > 0$ for $z$ in $\Delta$ and $0 < \theta_1 < \theta_2 < \cdots < \theta_{2p} < 2\pi$.

The class $\hat{C}(p,q,\alpha_1, \ldots, \alpha_{p-q})$ is defined to be the class of functions $f(z)$ such that $zf'(z)$ has the representation (1.6) except that no requirement that the $\theta_i$ all be different is made. The following lemma is then obvious.

**Lemma 1.** $\hat{C}(p,q,\alpha_1, \ldots, \alpha_{p-q})$ is dense in $\hat{C}(p,q,\alpha_1, \ldots, \alpha_{p-q})$.

We let $D(p,q)$ be the class of functions $f(z)$, analytic in $\Delta$, with $f(z) = z^q + \sum_{n=q+1}^{\infty}a_nz^n$ for $z$ in $\Delta$ such that $zf'(z)$ is analytic on $|z| = 1$ with the exception of $2p$ simple poles on $|z| = 1$ and such that there exists a $\beta = \beta(f)$ so that $\text{Im} e^{i\beta\ell}(e^{-i\beta/z}f(e^{-i\beta/z}))$ changes sign exactly $2p$ times on $|z| = 1$ [i.e., at the poles of $zf'(z)$].

**Theorem 2.** Every function in $\hat{C}(p,q,\alpha_1, \ldots, \alpha_{p-q})$ is the uniform limit in compacta of functions in $D(p,q)$.

**Proof.** Let $f(z)$ be in $\hat{C}(p,q,\alpha_1, \ldots, \alpha_{p-q})$ and suppose that in the representation (1.6) the function $p(z)$ is analytic on $|z| = 1$ with $\text{Re} e^{i\delta p(z)} > 0$ for some $\delta$ on $|z| = 1$. We can write

\begin{equation}
e^{i\delta p(z)} = \frac{z^{p-q}e^{i\delta} \prod_{i=1}^{p-q}(-\alpha_i)}{\prod_{i=1}^{p-q}(z - \alpha_i)(1 - \overline{\alpha_i} z)} \cdot \frac{\prod_{s=1}^{2p}(1 - e^{-i\theta_s}z)}{z^p} zf'(z).
\end{equation}

It is easily seen that

\[\text{arg} \left( \frac{z^{p-q}e^{i\delta} \prod_{i=1}^{p-q}(-\alpha_i)}{\prod_{i=1}^{p-q}(z - \alpha_i)(1 - \overline{\alpha_i} z)} \right)\]

is constant on $|z| = 1$ and that $\text{arg} \prod_{s=1}^{2p}(1 - e^{-i\theta_s}z)/z^p$ is constant for $z$ on the arc between $e^{i\theta_1}$ and $e^{i\theta_{p+1}}$ and that the argument changes by $\pi$ as $z$ goes from the arc between $e^{i\theta_1}$ to $e^{i\theta_{p+1}}$ to the arc between $e^{i\theta_{p+1}}$ to $e^{i\theta_{p+2}}$. It follows then that there exists a line through the origin such that $zf'(z)$ lies on one side of the line for $z$ on the arc between $e^{i\theta_1}$ and $e^{i\theta_{p+1}}$ and lies on the other side of the line for $z$ on the arc from $e^{i\theta_{p+1}}$ to $e^{i\theta_{p+2}}$. Thus there exists a $\beta = \beta(f)$ such that $\text{Im} e^{i\beta}(e^{-i\beta/z}f(e^{-i\beta/z}))$ changes sign exactly $2p$ times on $|z| = 1$.

If in the representation (1.6), $p(z)$ is not analytic on $|z| = 1$, then for each $r < 1$ we let $g_r(z)$ be given by

\begin{equation}
g_r(z) = \frac{\prod_{i=1}^{p-q}(z - \alpha_i)(1 - \overline{\alpha_i} z)}{z^{p-q} \prod_{i=1}^{p-q}(-\alpha_i)} \cdot \prod_{s=1}^{2p}(1 - e^{-i\theta_s}z)^{-1}.
\end{equation}
By the first part of the proof, \( g_r(z) \) is in \( D(p,q) \). By taking a sequence \( r_n \) converging to 1 we have \( g_n(z) \) converges uniformly to \( f(z) \) in compacta in \( \Delta \).

### III. Coefficient problem.

**Theorem 3.** Let \( f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n, \ z \) in \( \Delta \), be in \( D(p,q) \); then for \( n > p + 1 \),

\[
|a_n| < \sum_{k=q}^{p} \frac{2k(n+p)!}{(p+k)!(p-k)!(n-p-1)!(n^2-k^2)} |a_k|
\]

where \( a_q = 1 \).

**Proof.** We may obviously assume that \( \beta = 0 \). With this assumption the proof follows the proof of Theorem 1 in [7] with \( z f'(z) \) identified with the function in the statement of the theorem. We notice first that Lemma 1 of [7] goes through with \( f(z) \) in the statement of that lemma identified with \( z f'(z) \) where \( f(z) \) is in \( D(p,q) \). It is important to note that in the special case \( p = q = 1 \), the quantities \( \theta \) and \( \theta' \) in the proof of the theorem are such that \( z f'(z) \) has its two simple poles on \( |z| = 1 \) at \( e^{i\theta} \) and \( e^{i\theta'} \) and thus the two simple poles of \( e^{i\theta} z f'(e^{i\theta} z) \) are at \( e^{i\theta} \) and \( e^{-i\theta} \). Thus \( (1 + z^{-1} - 2\cos \nu) e^{i\theta} z f'(e^{i\theta} z) \) is analytic on \( |z| = 1 \). The proof of (1.12) on p. 411 in [7] can now be carried out. Let \( g(z) \) be defined as on p. 411 where we identify \( \phi(z) \) with \( z f'(z) \); then, as noted above, \( g(z) \) is analytic on \( |z| = 1 \). This is sufficient to carry out the proof. The proof starting on p. 413 can then be followed exactly giving us inequalities on the coefficients of \( z f'(z) \) which in turn give us (2.1).

**Remark 1.** There are two minor misprints on p. 415 of [7]. The summation in (4.19) should be \( \sum_{s=1}^{p-2} \) instead of \( \sum_{s=1}^{p-1} \) and in (4.21) the numerators in the square brackets should both be 1.

**Remark 2.** Equality cannot be attained in (2.1). An examination of the proof of (1.12) on p. 411 in [7] yields the fact that if equality occurs, then necessarily \( \nu \) is a multiple of \( \pi \) [i.e. we need \( |\sin k\nu|/|\sin \nu| = k \) for \( k = 1, 2, \ldots, n \)]. This would imply that \( z f'(z) \) has a multiple pole on \( |z| = 1 \), which cannot be the case for a function in \( D(p,q) \).

**Corollary 1.** If

\[
f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n,
\]

\( z \) in \( \Delta \), is in \( \hat{C}(p,q,\alpha_1, \ldots, \alpha_{p-q}) \), then the inequalities (2.1) are satisfied.

**Proof.** This follows immediately upon combining Lemma 1 and Theorems 2 and 3.

We let \( \mathcal{K}B \) denote the closed convex hull of any set \( B \) of functions analytic in \( \Delta \). The closed convex hulls and extreme points of a variety of classes of multivalent functions were determined in [4].

**Theorem 4.** \( \mathcal{K}\hat{C}(p,q,\alpha_1, \ldots, \alpha_{p-q}) = \mathcal{K}\hat{K}(p,q,\alpha_1, \ldots, \alpha_{p-q}) \).

**Proof.** We need only prove

\[
\mathcal{K}\hat{C}(p,q,\alpha_1, \ldots, \alpha_{p-q}) = \mathcal{K}\hat{K}(p,q,\alpha_1, \ldots, \alpha_{p-q})
\]
where \( B' \) denotes the class of derivatives of functions in any class \( B \).

Let \( X \) be the unit circle and \( \mathcal{P} \) the set of probability measures on \( X \). If \( zf'(z) \) is given by (1.4) then by the Herglotz representation for functions of positive real part we have

\[
(2.2) \quad p(z) = \int_X \frac{x + e^{-i\theta}z}{x - z} \, d\mu(x)
\]

and from [4] we have

\[
(2.3) \quad h(z) = \int_X \frac{z^p}{(1 - xz)^{2p+1}} \, dv(x)
\]

where \( \mu \) and \( v \) are in \( \mathcal{P} \).

Combining (1.4), (2.2), and (2.3) it can be seen that

\[
(2.4) \quad f'(z) = q \int_G \frac{||p_{-q}||^2(z - \alpha_i)(1 - \overline{\alpha}_i z)}{||p_{-q}||^2(-\alpha_i)} \frac{z^{q-1}(1 - xz)}{(1 - yz)^{2p+1}} \, d\mu(x, y)
\]

where \( G = X \times X \) and \( \mu \) is a probability measure on \( G \). Letting \( g'(z) \) be the kernel function in (2.4) we see that \( g'(z) \) has the representation (1.4) with \( h(z) = z^p/(1 - yz)^{2p} \) and \( p(z) = (1 - xz)/(1 - yz) \). It follows that (2.4) with \( \mu \) varying over all probability measures on \( G \), gives \( \mathcal{K}'(p, q, \alpha_1, \ldots, \alpha_{p-q}) \).

The representation formula for \( \mathcal{K}'(p, q, \alpha_1, \ldots, \alpha_{p-q}) \) is like (1.4) except that \( h(z) \) is replaced by a function of the form \( z^p[1 - e^{-i\theta}z]^{-1} \) which is in \( S(p, p) \). It is thus seen that (2.4) also holds for \( f(z) \) in \( \mathcal{K}'(p, q, \alpha_1, \ldots, \alpha_{p-q}) \).

We note that if \( g(z) \) denotes the kernel function in (2.4) then \( g(z) \) is in \( \mathcal{K}(p, q, \alpha_1, \ldots, \alpha_{p-q}) \). Thus (2.4) with \( \mu \) varying over the probability measures on \( G \) also gives \( \mathcal{K}'(p, q, \alpha_1, \ldots, \alpha_{p-q}) \), thereby proving the theorem.

**Theorem 5.** If \( f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n \) is in \( \mathcal{K}'(p, q, \alpha_1, \ldots, \alpha_{p-q}) \), then for \( n > p + 1 \),

\[
(2.5) \quad |a_n| < \frac{2k(n + p)!}{(p + k)! (p - k)! (n - p - 1)! (n^2 - k^2)} b_k
\]

where \( a_q = 1 \) and \( b_k = \sup |a_k|, k = 1, 2, \ldots, p, \) the sup being taken over all functions in \( \mathcal{K}'(p, q, \alpha_1, \ldots, \alpha_{p-q}) \).

**Proof.** We first note that the \( b_k, k = 1, 2, \ldots, p, \) exist. This follows from the fact that if \( f(z) \) is in \( \mathcal{K}'(p, q, \alpha_1, \ldots, \alpha_{p-q}) \), then the coefficients of \( f'(z) \) are majorized by the coefficients of

\[
q ||p_{-q}||^2(z + |\alpha_i|)(1 + |\alpha_i|z) \frac{z^{q-1}(1 + z)}{(1 - z)^{2p+1}}
\]

[Note that the above function is not in general the derivative of a function in \( \mathcal{K}'(p, q, \alpha_1, \ldots, \alpha_{p-q}) \).]

By Corollary 1, inequalities (2.1) and, hence, (2.5) are satisfied in \( \mathcal{K}'(p, q, \alpha_1, \ldots, \alpha_{p-q}) \). It is then easily seen that (2.5) holds in \( \mathcal{K}(p, q, \alpha_1, \ldots, \alpha_{p-q}) \).

We remark that the result above clearly holds for the class $K(p,q, \alpha_1, \ldots, \alpha_{p-q})$. It is also clear that the linear methods used in this paper will not give the exact Goodman conjecture [1] but only the closely related inequality proven in Theorem 5.

References


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