A CHARACTERISATION OF DISCRETENESS FOR LOCALLY COMPACT GROUPS IN TERMS OF THE BANACH ALGEBRAS $A_p(G)$

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Abstract. The Banach algebra $A$ is said to have the bounded power property if for any $x \in A$, with $\|x\|_p = \lim_{n \to \infty} \|x^n\|^{1/n} < 1$, one has $\sup_n \|x^n\| < \infty$. It has been shown by B. M. Schreiber [9, Theorem (8.6)] that, if $G$ is a locally compact abelian group, then the Fourier algebra $A(G) = L^1(T)$ has the bounded power property, if and only if $G$ is discrete. We improve this result in the THEOREM. Let $G$ be an arbitrary locally compact group and $1 < p < \infty$. Then $A_p(G)$ has the bounded power property if and only if $G$ is discrete. Our proof, even for abelian $G$ and $p = 2$ (then $A_2(G) = A(G)$ is the usual Fourier algebra of $G$), is much simpler and entirely different from that of [9].

The bounded power property for $L^1(G)$ with usual convolution (as in [7]) was investigated thoroughly by B. Schreiber [9]. Among other results, Schreiber obtains [9, Theorem (8.6)] the following result: (S) If $G$ is locally compact abelian then $L^1(G)$ has the bounded power property if and only if $G$ is compact.

In fact Schreiber proves more than that. He shows, among many other results, that if $G$ belongs to a class $\mathcal{S}$ of locally compact groups, which contains all abelian and all compact groups and all groups, all of whose unitary irreducible representations are finite dimensional, then $L^1(G)$ has the bounded power property if and only if $G$ is compact and abelian. He conjectures this result to be true for any locally compact $G$.

Schreiber's result (S) can be restated as follows: If $G$ is abelian, $\Gamma = \hat{G}$, then $A(\Gamma)$ has the bounded power property iff $\Gamma$ is discrete.

C. Herz has introduced and studied (see [5], [6]) for any locally compact $G$ and any $1 < p < \infty$, the function algebras $A_p(G)$. In case $p = 2$, $A_2(G)$ coincides with the Fourier algebra $A(G)$ studied extensively by P. Eymard [2] and in case $G$ is also abelian it coincides with the usual Fourier $A(G) = L^1(\Gamma)^\wedge$. We prove in this paper the following:

Theorem A. Let $G$ be any locally compact group and $1 < p < \infty$. Then $A_p(G)$ has the bounded power property iff $G$ is discrete.

We should point out that if $G$ is discrete, then $A_p(G)$ is a semisimple Banach algebra with discrete maximal ideal space, hence this part of Theorem A follows immediately from Corollary (2.3) of Schreiber [9, p. 408]. We give however, for the sake of completeness, an immediate proof of this part too.

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We have the following additional remarks:

(1) Schreiber’s proof of (S) uses: P. J. Cohen’s idempotent theorem, P. J. Cohen’s theorem on homomorphisms of group algebras which are implemented by piecewise affine maps ([1], or [8, Theorem 4.1.3, p. 78]) (both, in the proof of Lemma (6.1) of [9, pp. 415–416]), and J. E. Gilbert’s [4] and B. M. Schreiber’s [10] result on the representation of closed sets in the coset ring (in the proof of Theorem (6.2) [9, p. 416] which in turn is used in Lemma (8.5), hence in [9, Theorem (8.6), p. 429]). Our proof of Theorem A is simple and when restricted to abelian G and \( p = 2 \) (i.e. to \( A(G) \)) uses only basic facts taught in any first course in harmonic analysis.

(2) Our Theorem A does not throw any light on Schreiber’s conjecture mentioned above, but only reduces to the statement (S) in case \( p = 2 \) and G is abelian.

We express our thanks to M. Cowling for pointing out to us the paper [6] by C. Herz.

Notations. \( \lambda \) will denote a left Haar measure on G and we follow the basic notations of [7]. In particular, the \( L^p(G) \) norm will be denoted by 
\[
\|f\|_p = \left( \int |f|^p \, d\lambda \right)^{1/p}
\]
If \( 1 < p < \infty \) let \( p' \) be defined by \( 1/p + 1/p' = 1 \). For the algebra \( A_p(G) \) we follow C. Herz [5]. In particular, \( f \in A_p(G) \) iff \( f = \sum v_n \ast \tilde{u}_n \) (absolutely and uniformly convergent sum) with \( \sum \|v_n\|_p \|\tilde{u}_n\|_{p'} < \infty \). \( \|f\|_{A_p} \) will denote the infimum of these last sums over all such representations of \( f \). Herz proves in [5], among other results, that \( (A_p(G), \|\cdot\|_{A_p}) \) is a Banach algebra with respect to pointwise multiplication whose maximal ideal space is \( G \) (and which is a regular, tauberian algebra of functions on \( G \) [5, pp. 100–102]). Clearly \( \|f\|_{A_p} \leq \|f\|_{L^p(G)} \) for \( f \in A_p(G) \).

If \( S \subseteq G \), \( 1_S \) is defined by \( 1_S(x) = 1 \) if \( x \in S \) and equals 0 at all other \( x \in G \).

Proposition 1. Let \( G \) be a second countable locally compact group and \( 1 < p < \infty \). If \( A_p(G) \) has the bounded power property then \( G \) is discrete.

Proof. Let \( V \) be a symmetric relatively compact neighborhood of \( e \) the unit of \( G \). Consider the function
\[
\phi_V(x) = 1_V * 1_{\hat{V}}(x) = \int 1_V(x^{-1}y)1_V(y) \, dy = \lambda(xV \cap V).
\]
Then \( 0 \leq \phi_V(x) \leq \lambda(V) = \phi_V(e) \). Since
\[
\phi_V(e) \leq \|\phi_V\|_{A_p} \leq \|1_V\|_p \|1_V\|_{p'} = \lambda(V)
\]
one has that \( \phi_V(e) = \|\phi_V\|_{A_p} \). Let \( \psi_V = \lambda(V)^{-1} \phi_V \). Then
\[
\|\psi_V\|_{A_p} = 1 = \psi_V(e) \quad \text{and} \quad 0 \leq \psi_V(x) < 1.
\]
Moreover, \( \psi_V = 0 \) off \( V^2 \) since so does \( \phi_V \). It follows that \( V_1 = \{ x : \psi_V(x) > 0 \} \subseteq V^2 \) and, in particular, that for any neighborhood \( U \) of \( e \) there exists a relatively compact open neighborhood \( V_1 \) of \( e \) such that \( V_1 \subseteq U \) and for some \( \psi \in A_p(G) \) with \( 0 < \psi < 1 \), \( \|\psi\| = 1 = \psi(e) \) one has \( V_1 = \{ x : \psi(x) > 0 \} \).

Let \( \sim \) be a neighborhood base at \( e \) in \( G \) consisting of sets \( V_1 \) with this property.
Let $K$ be any closed set in $G$ and $W = G \setminus K$. Then, for every $a \in W$ there is some $V \in \mathcal{V}$ such that $aV \subset W$. Thus $W = \bigcup_{a \in W} a^\circ V$ where $V_n \in \mathcal{V}$ and $a_n \in W$ [11, p. 49]. Let $\psi_n = \psi_{a_n}$ be corresponding functions in $A_p(G)$ for $V_n$ and let $\psi = \sum_{n=1}^\infty \|a_n\|_p \psi_n$ where $(\lambda_f)(x) = f(a^{-1}x)$. $\psi \in A_p(G)$, since $\|a_n\|_p \psi_n \|_p = 1$. Clearly $0 < \psi < 1$, $\psi(x) > 0$ for all $x \in W$ and $\psi(x) = 0$ for all $x \in K$. We have shown that for any closed $K \subset G$ there exists $\psi \in A_p(G)$ such that $0 < \psi < 1$ and $K = \psi^{-1}(0)$.

From now on, let $K \subset G$ be compact, nowhere dense, and such that $\lambda(K) > 0$. If $G$ is nondiscrete, such $K$ exists. Let $\psi \in A_p(G)$ be such that $\lambda = \psi^{-1}(0)$ and $0 < \psi < 1$. Let $U \in A_p(G)$ be such that $0 < U < 1$ and $U(x) = 1$ for all $x \in K$. (Take as usual $U = \lambda(V)^{-1}[1_{KV} \cdot 1_V]$ where $V$ is any relatively compact symmetric neighborhood of $e$.) Then $\phi = U(1 - \psi) = U - U\psi \in A_p(G)$. Moreover $\{x; \psi(x) = 1\} = \{x; U(x) = 0\} = K$, i.e. $\lambda^{-1}(1) = K$. Our assumption implies now that $\sup \|\phi^n\| < \infty$. Thus, $\{\phi^n; n \geq 1\}$ is a $w^*$ compact subset of the Banach algebra $B_p(G)$, which is the Banach space dual of the normed space $L^1(G)$ with the norm $QF_p$ (which is stronger than $PF_p$, the norm on $L^1(G)$ acting as convolution operators on $L^p(G)$). In case $G$ is amenable, $B_p(G)$ is the dual of $L^1(G)$ with the $PF_p$ norm. See C. Herz [6, Proposition 2 and the remarks thereafter]. Thus there exists some $\psi \in B_p(G)$ (which consists only of bounded continuous functions [Herz [6, Proposition 3]]) such that $\int \phi^n f(x) dx \to \int \psi f(x) dx$ for all $f \in L^1(G)$. But

$$
\lim_{n \to \infty} \int \phi^n f(x) dx = \int 1_K(x) f(x) dx \text{ for all } f \in L^1(G).
$$

Hence $1_K(x) = \psi(x)$ a.e. Thus $\psi^2(x) = \psi(x)$ a.e. and since $\psi(x)$ is continuous, $\psi^2(x) = \psi(x)$ for all $x$. Thus $\psi(x) = 1_K(x)$ for some open and closed $K_1 \subset G$ such that $1_K = 1_{K_1}$ a.e. But $K_1 \sim K$ is open and $\lambda(K_1 \sim K) = 0$. Thus $K_1 \subset K$. Since $\lambda(K_1) = \lambda(K_1) > 0$, $K_1$ is nonvoid, which contradicts the fact that $K$ is nowhere dense.

**Remark.** If $G$ is abelian and $p = 2$, then $B_2(G) = M(\Gamma)^*$, and we used only the fact that $B_2(G)$ is a dual space (to $C_0(\Gamma)$) and its unit ball is $w^*$ compact.

**Theorem.** Let $G$ be any locally compact group, $1 < p < \infty$. If $A_p(G)$ has the bounded power property then $G$ is discrete.

**Proof.** Let $G_0 = \bigcup_{i} \mathcal{U}^n$ where $U$ is a symmetric relatively compact neighborhood of $e$ in $G$. Let $\phi_0 \in A_p(G_0)$ be such that $|\phi_0(x)| < 1$ for all $x$. Let $\phi(x) = \phi_0(x)$ for $x \in G_0$ and $\phi(x) = 0$ for other $x \in G$. Then by Herz [5, p. 106] $\phi \in A_p(G)$ and $\|\phi_0^n\|_{A_p(G_0)} < \|\phi^n\|_{A_p(G)}$. We can hence (and shall) assume that $G$ is compactly generated (since if $G_0$ is discrete so is $G$). Assume that $G$ is not discrete and let $V_n$ be a sequence of relatively compact neighborhoods of $e$ such that $\lambda(V_n) \to 0$ and let $K \subset \bigcap_{n} \mathcal{U}^n$ be a compact normal subgroup such that $G_1 = G/K$ is separable metric. Let $\phi_1 \in A_p(G_1)$ be such that $|\phi_1(x^1)| < 1$ for all $x^1 \in G_1$ and let $\phi(x) = \phi_1(x^1)$ where $x^1 \in G_1$ represents the coset $xK$. Then, by Herz [5, p. 106] $\|\phi^n\|_{A_p(G_1)} \leq \|\phi^n\|_{A_p(G)}$. Clearly, $|\phi(x)| < 1$ for all $x$, hence, $\sup \|\phi^n\|_{A_p(G)} < \infty$. This shows that $A_p(G)$ has the bounded power property. By Proposition 1, $G_1$ is discrete, which implies that $K$ is open. But $\lambda(K) = 0$ which cannot be. Hence,
there does not exist in $G$ a sequence of neighborhoods of $e$, $V_n$, such that $\lambda(V_n) \to 0$, i.e. $G$ is discrete.

For the sake of completeness we give a short easy proof of the converse of Theorem 1. This converse is due to B. M. Schreiber [9, Corollary 2.3, p. 408].

**Proposition 2.** Let $G$ be discrete. Then $A_p(G)$ has the bounded power property.

**Proof.** Let $f_i \in A_p(G)$ have finite support and $|f_i(x)| < 1$ for all $x$. Then $f_i = \sum f_i(a_i)\delta_{a_i}$ where $\delta_{a_i}$ is the unit mass at $a_i$ (hence $\|\delta_{a_i}\|_{A_p} = \|\delta_e\|_{A_p} = 1$). Then

$$\|f^n_i\|_{A_p} = \left\| \sum_{i=1}^k f_i(a_i)^n \delta_{a_i} \right\|_{A_p} \leq k.$$  

Hence $\sup_{n} \|f^n_i\|_{A_p} < \infty$. If $f \in A_p(G)$ is arbitrary with $|f(x)| < 1$, let $f_1(x) = f(x)1_F(x)$ where $F = \{x; |f(x)| = 1\}$ and $f_2 = f(1 - 1_F)$. Then $\sup_x |f_2(x)| = \alpha < 1$, so $\|f_2^n\|_{A_p}^{1/n} \to \alpha < 1$, i.e. $\|f_2^n\|_{A_p} \to 0$. Now

$$\|f^n\|_{A_p} = \|f_1^n + f_2^n\| \leq \|f_1^n\| + \|f_2^n\|.$$  

Since $\|f_1^n\|$ is bounded by the first part, if follows that $\sup_{n} \|f^n\| < \infty$.

**References**


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