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TIETZE-TYPE THEOREMS ON MONOTONE INCREASING SETS

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ABSTRACT. The Tietze theorem on convex sets is generalized to monotone increasing sets and strictly monotone increasing sets, which include convex sets as a special case. The main theorem is that a closed connected set in E_2 is monotone increasing if and only if it is locally monotone increasing. A similar result is proved for strictly monotone increasing sets.

1. **Introduction and preliminaries.** In 1928 Tietze [3] proved that a closed connected set in E_n is convex if and only if it is locally convex. Klee [1] generalized this theorem to a topological linear space. We will prove that a closed connected set in E_2 is monotone increasing if and only if it is locally monotone increasing. A similar result is proved for strictly monotone increasing sets.

Let E_2 be a two-dimensional Euclidean space with the rectangular coordinate axes. The closure, interior, boundary, and convex hull of a set S in E_2 are denoted by $\operatorname{cl} S$, $\operatorname{int} S$, $\operatorname{bd} S$, $\operatorname{conv} S$, respectively. If x and y are distinct points, then xy denotes the closed line segment joining x and y and $\operatorname{intv} xy$ denotes the relative interior of xy. For two distinct points x and y in E_2 not lying on a vertical line, let $m(x,y) = m(L(x,y))$ denote the slope of the line $L(x,y)$ through x and y. If x and y are two points in E_2 not lying on a vertical or horizontal line, the line segment xy determines two closed convex triangles having xy as the hypotenuse and each of the remaining sides parallel to one of the axes. The triangle lying below xy is denoted by $T(x,y)$, called the lower triangle determined by x and y. A convex arc $C(x,y)$ joining x and y is called a monotone increasing arc if $m(x,y) > 0$ and $C(x,y) \subset T(x,y)$.

DEFINITION 1. A set S in E_2 is monotone increasing if for each pair of distinct points $x \in S$, $y \in S$, it is true that

(1) there exists a monotone increasing arc $C(x,y)$ in S joining x and y if $m(x,y) > 0$; and

(2) $xy \subset S$ if x and y are on a vertical line or if $m(x,y) \leq 0$.

We note that the arc $C(x,y)$ in (1) of Definition 1 above can contain at most one horizontal line segment since $C(x,y)$ is a convex arc and $C(x,y) \subset T(x,y)$.

DEFINITION 2. A set S in E_2 is strictly monotone increasing if for each pair of distinct points $x \in S$, $y \in S$, it is true that

(1) if $m(x,y) > 0$, there exists a monotone increasing arc $C(x,y)$ joining x

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and y in S where $C(x,y)$ contains no horizontal line segment; and

(2) $xy \subset S$ if x and y are on a vertical line or if $m(x,y) \leq 0$.

DEFINITION 3. A set S in E_2 is locally (strictly) monotone increasing at a point $z \in S$ if there exists a neighborhood N of z such that $N \cap S$ is (strictly) monotone increasing. A set S is locally (strictly) monotone increasing if it is locally (strictly) monotone increasing at each of its points.

We observe that if $S \subset E_2$ is an open monotone increasing set, then S is strictly monotone increasing. To see this, let x and y be arbitrary distinct points of S. If x and y lie on a vertical line or if $m(x,y) \leq 0$, then $xy \subset S$. Suppose $m(x,y) > 0$, and suppose that $C(x,y)$ is a monotone increasing arc joining x and y in S with y lying in the upper half-plane bounded by the horizontal line through x. Let xu be the longest horizontal line segment contained in $C(x,y)$. Since S is open, for each $z \in xu$, there exists an open set N_z of z such that $N_z \subset S$. The compactness of the line segment xu implies the existence of a finite collection of these open sets, say $N_{z_1}, \ldots, N_{z_n}$, with xu contained in $N_{z_1} \cup \cdots \cup N_{z_n} \subset S$. Without loss of generality, we assume that $u \in N_{z_n}$. Now we choose a point w in $N_{z_n} \cap C(u,y)$, where $C(u,y)$ is the subarc of $C(x,y)$ joining u and y, such that $w \neq u$, $w \neq y$ and $xw \subset N_{z_1} \cup \cdots \cup N_{z_n}$. The set $xw \cup C(w,y)$, where $C(w,y)$ is the subarc of $C(x,y)$ joining w and y, is a monotone increasing arc containing no horizontal line segment and joining x and y in S. This proves that S is strictly monotone increasing.

To prove our characterization theorems, we apply a result on stripwise subconvex sets [2]. The relevant definitions and theorem are stated below.

DEFINITION 4. A set S in E_2 is stripwise subconvex if for each pair of distinct points x and y in S there exists a convex arc $A(x,y)$ in S joining x and y such that

(1) $A(x,y) = xy$ if x and y are on a vertical line; and

(2) if x and y are not on a vertical line, then $A(x,y)$ lies in the closed lower half-plane bounded by the line $L(x,y)$ and $A(x,y)$ lies in the convex hull of the vertical lines through x and y respectively. Such an arc $A(x,y)$ is called a stripwise subconvex arc.

DEFINITION 5. A set S in E_2 is locally stripwise subconvex if for each $z \in S$ there exists a neighborhood N of z such that $N \cap S$ is stripwise subconvex.

THEOREM 1. *A closed connected set S in E_2 is stripwise subconvex if and only if it is locally stripwise subconvex* [2].

We note that a monotone increasing set is stripwise subconvex but the converse is not true in general.

If $S \subset E_2$ is a closed set and two points x and y in S can be joined by a stripwise subconvex arc in S, then there exists a minimal stripwise subconvex arc $C_0(x,y)$ joining x and y in S in the sense that if $C(x,y)$ is any other stripwise subconvex arc joining x and y in S, it is true that $\operatorname{conv} C_0(x,y) \subset \operatorname{conv} C(x,y)$. The same assertion holds for monotone increasing arcs. To see this, first consider the case when $xy \not\subset S$. Let $\{C(x,y)\}$ be the collection of all stripwise subconvex (monotone increasing) arcs joining x and y in S, this collection is nonempty by hypothesis. Now set

$$C_0(x,y) = \operatorname{bd}\{\cap \operatorname{conv} C(x,y)\} - \operatorname{intv} xy$$

where the intersection is taken over all members of the collection. Since the set S is closed, we have $C_0(x,y) \subset S$. Clearly $C_0(x,y)$ is a minimal stripwise subconvex (monotone increasing) arc joining x and y in S. In case $xy \subset S$, then xy is the minimal arc required.

The theorem stated below will be useful in the following discussion. A more general form of this theorem can be found in Valentine [4].

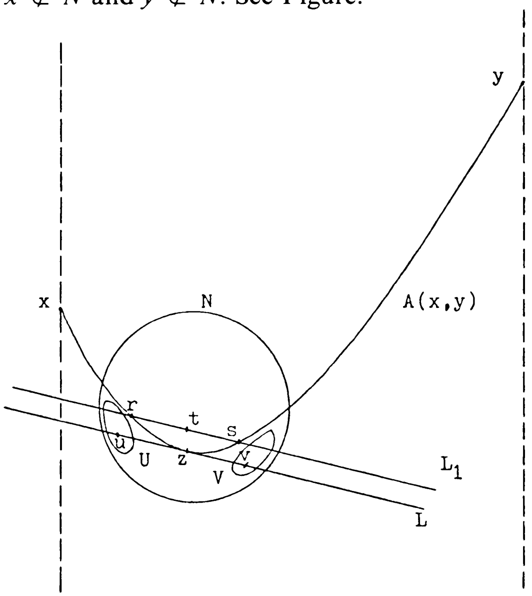
THEOREM 2. *Let S be a closed convex set in E_2. Each compact connected portion of the boundary of S which is not contained in a line segment contains an exposed point of S. If S is a line segment, it has two exposed points.*

2. The results.

THEOREM 3. *Let S be a closed connected set in E_2. The set S is monotone increasing if and only if it is locally monotone increasing.*

PROOF. The necessity is obvious. To prove the sufficiency, let x and y be arbitrary distinct points in S. Since S is locally monotone increasing, it is locally stripwise subconvex. By Theorem 1, S is stripwise subconvex. If x and y lie on a vertical line, then $xy \subset S$. Otherwise, let $A(x,y)$ be a stripwise subconvex arc joining x and y in S. We may assume that $A(x,y)$ is minimal. Now we consider the following cases according to the slope $m(x,y)$.

(a) $m(x,y) > 0$. If $A(x,y) = xy$, there is nothing to prove. Suppose that $A(x,y) \neq xy$, and without loss of generality we may assume that y lies in the upper half-plane bounded by the horizontal line through x. We will show that $A(x,y)$ is a monotone increasing arc joining x and y in S. Let z be an exposed point of $\operatorname{conv} A(x,y)$ on $A(x,y) - \{x,y\}$ and let L be a corresponding line of support to $\operatorname{conv} A(x,y)$ through z for which $L \cap \operatorname{conv} A(x,y) = \{z\}$. We wish to show that L has positive slope. Suppose on the contrary that $m(L) \leq 0$. Since S is locally monotone increasing, a neighborhood N of z exists such that $N \cap S$ is monotone increasing. We may choose N to be a convex neighborhood such that $x \notin N$ and $y \notin N$. See Figure.



FIGURE

Since $A(x,y)$ is compact, $\operatorname{conv} A(x,y)$ is compact. We can choose points u and v on L with $z \in \operatorname{intv} uv$, and neighborhoods U of u and V of v such that $U \subset N$, $V \subset N$, $U \cap \operatorname{conv} A(x,y) = \varnothing$, and $V \cap \operatorname{conv} A(x,y) = \varnothing$. The fact $A(x,y) \neq xy$ implies that $\operatorname{int} \operatorname{conv} A(x,y) \neq \varnothing$, thus

$$\operatorname{cl} (\operatorname{conv} A(x,y)) = \operatorname{cl} (\operatorname{int} \operatorname{conv} A(x,y)).$$

Hence any neighborhood of z contains an interior point of $\operatorname{conv} A(x,y)$. Choose a point $t \in \operatorname{int} \operatorname{conv} A(x,y) \cap N$ sufficiently close to z so that there exists a line L_1 through t parallel to L with $L_1 \neq L$ such that $L_1 \cap U \neq \varnothing$ and $L_1 \cap V \neq \varnothing$. Therefore $L_1 \cap \operatorname{conv} A(x,y)$ contains no points of U and V and $L_1 \cap \operatorname{conv} A(x,y) \subset N$. Let $r\,s$ denote $L_1 \cap \operatorname{conv} A(x,y)$ where r and s are boundary points of $\operatorname{conv} A(x,y)$ and, therefore, points of $A(x,y)$. The line segment rs is nondegenerate since $t \in \operatorname{intv} rs$. We have $m(r,s) = m(L_1) = m(L) \leq 0$ since L_1 is parallel to L. Since $N \cap S$ is monotone increasing and r and s lie in $N \cap S$ with $m(r,s) \leq 0$, it follows that $rs \subset S$. Now set $B(x,y) = A(x,r) \cup rs \cup A(s,y)$ where $A(x,r)$ and $A(s,y)$ are the subarcs of $A(x,y)$ joining x to r and s to y respectively. Since z is an exposed point of $\operatorname{conv} A(x,y)$ and $z \in A(r,s)$ where $A(r,s)$ is the subarc of $A(x,y)$ joining r and s, we have $A(r,s) \neq rs$. The arc $B(x,y)$ is a stripwise subconvex arc joining x and y in S. The existence of $B(x,y)$ contradicts the minimality of $A(x,y)$ since $\operatorname{conv} A(x,y)$ is not contained in $\operatorname{conv} B(x,y)$. Hence L must have positive slope.

If $A(x,y)$ is not contained in the closed upper half-plane bounded by the horizontal line through x, let w be the point of intersection of $A(x,y)$ and the horizontal line through x. Thus $A(x,w) - \{x,w\}$ lies in the open lower half-plane bounded by the horizontal line through x where $A(x,w)$ is the subarc of $A(x,y)$ joining x to w. Theorem 2 and this fact imply that there exists an exposed point b of $\operatorname{conv} A(x,w)$ on $A(x,w) - \{x,w\}$ with a line of support K to $\operatorname{conv} A(x,w)$ through b such that $K \cap \operatorname{conv} A(x,w) = \{b\}$ and $m(K) \leq 0$. Clearly the point b is an exposed point of $\operatorname{conv} A(x,y)$ and the line K is a line of support to $\operatorname{conv} A(x,y)$ through b such that $K \cap \operatorname{conv} A(x,y) = \{b\}$. The fact $m(K) \leq 0$ contradicts what we proved above. Hence $A(x,y)$ lies in the closed upper half-plane bounded by the horizontal line through x. But $A(x,y)$ is stripwise subconvex, thus $A(x,y)$ must lie in the lower triangle $T(x,y)$ and, hence, $A(x,y)$ is a monotone increasing arc joining x and y in S.

(b) $m(x,y) < 0$. For z and L described in (a), L has positive slope. From Theorem 2 and the fact $m(x,y) < 0$, it follows that $A(x,y) = xy$.

(c) $m(x,y) = 0$. As in (a) we conclude that $A(x,y)$ must lie in the closed upper half-plane bounded by the horizontal line through x. Consequently $A(x,y) = xy$.

We conclude that S is a monotone increasing set.

THEOREM 4. *Let S in E_2 be a closed connected set. The set S is strictly monotone increasing if and only if it is locally strictly monotone increasing.*

PROOF. The necessity is obvious. To prove the sufficiency, let x and y be arbitrary distinct points in S. By Theorem 3, S is monotone increasing. Hence $xy \subset S$ if x and y are on a vertical line or if $m(x,y) \leq 0$. If $m(x,y) > 0$, then there exists a monotone increasing arc $C(x,y)$ joining x and y in S. Since S is closed, we may take $C(x,y)$ to be minimal. We only need to show that $C(x,y)$ contains no horizontal line segment. Assume that y lies in the upper half-plane bounded by the horizontal line through x. Suppose xz is the longest horizontal line segment of $C(x,y)$. Since S is locally strictly monotone increasing at the point z, there is a neighborhood N of z such that $N \cap S$ is strictly monotone increasing. If $x \neq z$, there exists a point $s \in N \cap \operatorname{intv} xz$, and a point $t \in N \cap C(z,y)$ with $t \neq z$ and $t \neq y$ where $C(z,y)$ is the subarc of $C(x,y)$ joining z and y. The points s and t lie in $N \cap S$, and $m(s,t) > 0$, therefore there exists a monotone increasing arc $D(s,t)$ in $N \cap S$ such that $D(s,t)$ contains no horizontal line segment. Let

$$D_0(s,t) = \operatorname{bd}\{\operatorname{conv} D(s,t) \cap \operatorname{conv} C(s,t)\} - \operatorname{intv} st$$

where $C(s,t)$ is the subarc of $C(x,y)$ joining s and t. It is obvious that $D_0(s,t) \subset S$. Set $C_0(x,y) = xs \cup D_0(s,t) \cup C(t,y)$, where $C(t,y)$ is the subarc of $C(x,y)$ joining t and y. The arc $C_0(x,y)$ is clearly convex and it is a monotone increasing arc joining x and y in S. But $\operatorname{conv} C(x,y)$ is not contained in $\operatorname{conv} C_0(x,y)$ since $D(s,t)$ and $D_0(s,t)$ contain no horizontal line segment. The existence of $C_0(x,y)$ contradicts the minimality of $C(x,y)$. Thus $x = z$ and $C(x,y)$ contains no horizontal line segment. Hence S is a strictly monotone increasing set.

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