SOME APPLICATIONS OF LANDWEBER-NOVIKOV OPERATIONS

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Abstract. Previous results on the characteristic numbers of $Sp$-manifolds are extended in three different ways. I. It is shown that the primitive symplectic Pontrjagin class evaluated on a $4(2^r - 1)$ dimensional $Sp$-manifold always gives a number divisible by 8. This forms an analogue to a well-known result of Milnor concerning $U$-manifolds. II. It is shown that some of the results of Floyd as well as an analogue of the previous result can be obtained for 'pseudo-symplectic' manifolds. III. Results are generalised to $(Sp,fr)$ manifolds.

1. $4(2^r - 1)$ dimensional $Sp$-manifolds. Let $s_\pi(p)[M]$, $\pi$ a partition of $n = n(\pi)$, $M$ a $4n(\pi)$ dimensional stably symplectic manifold, denote the normal symplectic Pontrjagin number of $M$ corresponding to the $\pi$-symmetrised polynomial in a system of indeterminates for which the symplectic Pontrjagin classes are the elementary symmetric polynomials. Throughout this section we will set $k = 2^r - 1$ and $M$ will denote a $4k$ dimensional stably symplectic manifold.

Theorem 1.1. $8 \mid s_{(k)}(p)[M]$.

Remarks. 1. The unitary analogue, $2 \mid s_{(k)}(c)[N]$, $N$ stably unitary is well known; it could be proven by the techniques used below.

2. The techniques of [3] are not adequate by themselves to prove Theorem 1.1.

Proof. Actually we will prove slightly more: Let $\pi$ be any partition of $k$ all of whose parts are themselves integers of the form $2^s - 1$. Then $8 \mid s_\pi(p)[M]$.

If $\pi = (a_1, \ldots, a_r)$, let $D(\pi) = \prod_i [(2a_i + 2)!/2]$. Well known fact. $2 \mid \sum_{n(\pi)=k} (s_\pi(p)[M]/D(\pi))$. This is the 'Todd genus' relation of Stong [4] who put things in an 'abnormal' form; using normal rather than tangential numbers makes computation manageable. In particular, we can see that for a fixed $k$ the denominators $D(\pi)$ with maximal number of factors of 2 will be just those for which all parts of $\pi$ are of the form $2^s - 1$.

By Proposition 4 of [3] it is automatic that $4 \mid s_\pi(p)[M]$ for all $\pi$, $n(\pi) = k$. If we can show that $8 \mid s_\pi(p)[M]$ whenever $\pi = (a_1, \ldots, a_r)$, $n(\pi) = k$, $r > 1$
and all \( a_i \) of the form \( 2^i - 1 \), then it will follow from the above that \( 8 \mid s_{(k)}(p)[M] \).

Now assume inductively that Theorem 1.1 holds in dimensions less than \( 4k \); by [4] it certainly holds in dimensions 4 and 12. Let \( \vec{a} = (a_1, \ldots, a_r) \) be a partition of \( k \) with \( r > 1 \) and all \( a_i \) of the form \( 2^i - 1 \). Since \( \vec{a} \) is a partition of an odd number into odd parts there is a number \( k' \) which occurs exactly \( f \) times as a part of \( \vec{a} \), \( f \) odd. Let \( \vec{a}' \) denote the partition obtained from \( \vec{a} \) by deleting one occurrence of \( k' \). Let \( S(\vec{a}') \) denote the symplectic Landweber-Novikov operation corresponding to \( \vec{a}' \). Then from the results of [1] on the action of such operations,

\[
(1.2) \quad s_{(k')}(p)[S(\vec{a})M] = f s_{\vec{a}}(p)[M] + \sum_{\vec{a}} a(\vec{a}, \vec{a}, \vec{a}') s_{\vec{a}}(p)[M],
\]

where the summation on the right runs through all partitions \( \vec{a} \) obtained by adding \( k' \) to one of the parts of \( \vec{a}' \), and the coefficients \( a(\vec{a}, \vec{a}, \vec{a}') \) are integers which are in fact even as a consequence of the fact that the parts of \( \vec{a}' \) are all of the form \( 2^i - 1 \). Then by Proposition 4 of [3] (and since \( n(\vec{a}) \) is odd), 8 divides the summation term. But by the inductive hypothesis, 8 divides the left side of (1.2). Our assertion and the theorem then follow from the oddness of \( f \).

2. Pseudo-symplectic manifolds. We call a \( U \)-manifold pseudo-symplectic if some nonzero multiple of its class in \( MU_* \) is in the image of \( MSp_* \); this will be the case if and only if every Chern number of the manifold involving an odd Chern class vanishes. Let \( Ps_* \) be the subring of \( MU_* \) consisting of such classes. Let \( j, p, d \) be the maps in the cofibration sequence of spectra

\[
MSp \to MU \to MU/MSp \to SMSp.
\]

There is a well-defined map \( h_{Ps}^*: Ps_* \to H_*(MSp) \) obtained by restricting the Hurewicz homomorphism \( h^*_{MU}: MU_* \to H_*(MU) \) to \( Ps_* \) and then composing with \( j_*^{-1} \). We regard the symplectic Pontrjagin numbers as defined on \( Ps_* \).

Note that \( Im h_{Ps} \subset Im h_{Ps} \subset H_*(MSp) \) (inclusions strict) and that \( Im h_{Ps}^*/Im h_{Ps}^* \) gives the torsion elements of \( Im p_* \) in the \( (MU,MSp) \) exact bordism sequence.

**Lemma 2.1.** Let \( S(\pi): MSp \to S^{4n(\pi)}MSp \) be a symplectic Landweber-Novikov operation. Then we can find some \( U \)-bordism operation \( T: MU \to S^{4n(\pi)}MU \) such that \( T \circ j = S^{4n(\pi)}j \circ S(\pi) \).

**Proof.** Treat \( S^{4n(\pi)}j \circ S(\pi) \) as a class in \( MU^{4n(\pi)}(MSp) \). Now \( d_* (S^{d(\pi)}j \circ S(\pi)) = 0 \) in \( MU^{4n(\pi)+1}(MU/MSp) \) (since that group is trivial), so by exactness there must exist \( T \in MU^{4n(\pi)}(MU) \) such that \( j_* (T) = S^{d(\pi)}j \circ S(\pi) \).

This ‘compatibility’ lemma implies that \( Im h_{Ps}^* \) is closed under the action of the symplectic Landweber-Novikov operations.

**Theorem 2.2.** Let \( M \) be a 4k dimensional pseudo-symplectic manifold. Then

(i) \( 2 \mid s_{n}(p)[M] \) if \( n(\pi) \) is odd or if \( \pi = (2^i) \);

(ii) \( 4 \mid s_{k}(p)[M] \) if \( k = 2^i - 1 \).

**Remark.** Floyd first studied pseudo-symplectics (they are the ‘related
manifolds' of the title of [2]) and part (i) was proved by him by rather different methods.

Proof. Exactly as for symplectics in [3] except that one has weaker low-dimensional divisibility properties to feed into the machinery so that statements involving 4 become statements involving 2 while those involving 2 become vacuous. By the same token, part (ii) is done on the model of Theorem 1.1 above.

3. (Sp, fr) manifolds. Let \( h^*_\text{Sp,fr} : \text{MSp}/fr^\# \to H_*^{\text{fr}}(\text{MSp}/fr) \) be the Hurewicz map for \( \text{MSp}/fr \), the spectrum representing (Sp,fr) bordism. We wish to obtain divisibility conditions on characteristic numbers of (Sp,fr) manifolds. One would expect to use a compatibility lemma which showed that \( \text{Im } h^*_\text{Sp,fr} \) is closed under the action of symplectic Landweber-Novikov operations, find some 'starting' conditions and proceed as with the symplectic and pseudo-symplectic cases.

Actually something happens which makes our work easier (and our results stronger). If \( n(\pi) > 0 \) then \( S(\pi) \) can be lowered to a map \( S(\pi)' : \text{MSp}/fr \to S^{4n(\pi)} \text{MSp} \) so that \( S(\pi) \) actually sends the (Sp,fr) classes into full-fledged Sp-classes. Thus all the divisibility conditions of [3] hold equally for (Sp,fr) manifolds except in the starting dimensions:

**Theorem 3.1.** Let \( M \) be a \( 4k(\text{Sp,fr}) \) manifold. Then

(i) \( 4 | s_n(p)[M] \) if \( n(\pi) > 1 \) and odd or if \( \pi = (2^j), j > 1 \);

(ii) \( 2 | s_n(p)[M] \) if \( n(\pi) > 2 \) and \( \equiv 2 (4) \) or if \( \pi = (2^j,2^i), j > 1 \).

The proof of Theorem 1.1 does not carry over to the (Sp,fr) case.

**Bibliography**


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