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EQUIVALENCE OF CERTAIN DISCONTINUOUS FUNCTIONS UNDER CLOSURE

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ABSTRACT. We show that no distinction can be made between the closure of a Darboux graph and the closure of a connected graph, and under certain conditions, we show the closure of a Darboux function is the closure of an almost continuous function. In showing this, a construction process is given to show how to turn certain non-almost continuous functions into almost continuous functions.

Similarities and differences between Darboux, connected and almost continuous functions have been of recent interest. For instance, each real function is the pointwise limit of a sequence of Darboux, connected, or almost continuous functions and each real function is the sum of two Darboux, connected, or almost continuous functions [2], [9]. Also, each real function f is equal to a connected, respectively Darboux, function h except on a first category, F_σ set with measure 0 and each function has the property that the set of points at which it is Darboux or connected is a G_δ set [2], [10]. One obvious difference is the fact that a Darboux function need not have a fixed point while a connected or almost continuous function must have a fixed point [11]. There are many examples of Darboux functions which are not connected, and in [5], Jones and Thomas give an interesting example of a connected function which is not almost continuous. In pointing out similarities of the above classes of functions under closure, we examine the Jones-Thomas example and give a construction process to make it almost continuous. The author wishes to thank Professor Harvey Rosen for many helpful conversations during the preparation of this paper. Also, the author is indebted to the referee for several helpful suggestions for shortening the proofs of this paper in order to make them more readable.

Preliminaries. No distinction will be made between a function and its graph and unless otherwise stated, all functions will be from $I = [0, 1]$ into I. If M is a subset of I^2 then M_K denotes those points of M which have X-projection in K where K is a subset of I. We denote the vertical line through the point $(x, 0)$ by l_x. If f is a function from I to I then $f(+)$ denotes the subset of I^2 consisting of all ordered pairs (x, y) where x is in I and $y > f(x)$. We

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define $f(-)$ analogously. By $\overline{M}$, we will mean the closure of M. A function f is said to be a *Darboux* function if $f(C)$ is connected whenever C is a connected subset of the domain of f. A function f is said to be *connected* if f is a connected subset of I^2. If each open set containing a function f also contains a continuous function with the same domain as f, then f is said to be *almost continuous* [11]. The function g is said to be a *Baire class* 1 function if and only if g is the pointwise limit of a sequence of continuous functions. A function f defined over an interval I is said to be bilaterally dense-in-itself provided for each x in f, each open square, which has a vertical side bisected by x and whose X-projection is contained in I, contains infinitely many points of f [2]. All previously known results listed in the sequel will be labeled as properties. Our first result shows that no distinction can be made between the closure of a Darboux function and the closure of a connected function.

THEOREM 1. *Let f be a Darboux function on an interval I. Then there exists a connected function g such that $\bar{f} = \bar{g}$.*

PROOF. Denote by S the set of discontinuities of f and by T a countable dense subset of I such that the closure of f_T (taken in I^2) is $\bar{f}$. Let $A = (I - S) \cup T$. Then f_A is bilaterally dense-in-itself, because f is a Darboux function.

Denote by C the family of all planar continua K for which the X-projection of $K \cap \bar{f}$ is uncountable and K misses f_A. For each K in C, K meets uncountably many sets L_x where $L_x = l_x \cap \bar{f}$ and x is in $S - T$. Since f is a Darboux function and x is in $S - T$, then L_x is a nondegenerate closed interval [8]. Let $\Gamma(K)$ denote the family of all such L_x for each K.

Now for each K in C, there exist uncountably many L in $\Gamma(K)$ which are bilateral limit points of $\Gamma(K)$ (see Lemma 1 of [10]). Denote by $\Psi(K)$ the family of all such L.

Since C can be well ordered so that no term has uncountably many predecessors it follows by an induction argument that there exists a one-to-one function h from C into $\bigcup\{\Psi(K) : K \text{ in } C\}$ such that $h(K)$ is in $\Psi(K)$. Now let H denote the range of h and B denote the X-projection of H. For $h(K)$ in H with x equal to the X-projection of $h(K)$, we define $g(x)$ to be any point in the Y-projection of $(K \cap \bar{f} \cap l_x)$. For all x in I for which g is not defined by the previous process, we define $g(x) = f(x)$.

Since $f_T = g_T \subseteq g$ and $\bar{f} = \bar{f}_T$, we have $\bar{f} \subseteq \bar{g}$. Moreover, for x not belonging to B, we have $f(x) = g(x)$, and by construction, if z is in B, then $(z, g(z))$ is contained in $\bar{f} \cap l_z$. Hence $g \subseteq \bar{f}$ which implies $\bar{g} \subseteq \bar{f}$. Hence $\bar{g} = \bar{f}$.

Now g will be connected if no continuum M hitting both $g(+)$ and $g(-)$ misses g [4, Theorem 2]. Suppose then that M is a continuum meeting $g(+)$ and $g(-)$ and missing g. Now if we can show g is bilaterally dense-in-itself we are through, as can be shown as follows.

If g is bilaterally dense-in-itself, then according to Theorem 1 of [4] or arguments in [7], there exists a perfect set F contained in I such that $l_x \cap \bar{g} \cap M$ is not empty for each x in F. Hence the X-projection of $\bar{g} \cap M$, which is the X-projection of $\bar{f} \cap M$, is uncountable and M misses $g_A = f_A$, so that M is in C. Then, by construction, g must hit M and this is a contradiction.

Therefore, to complete the proof, we need only show g is bilaterally dense-in-itself. To see g is bilaterally dense-in-itself, note (1) any point of g_B is a bilateral limit point of f_S by construction. But since $\bar{f}_S \subseteq \bar{f}_T = \bar{f}$ and $f_T = g_T$, any point of g_B is a bilateral limit point of g_T. Also, (2) any point of g_{I-B} is a right (left) limit point of f_B or f_{I-B}. In the former case, $f_B \subseteq f_S$ and $\bar{f}_S \subseteq \bar{f}_T = \bar{g}_T = \bar{g}$, and in the latter case, $f_{I-B} = g_{I-B} \subseteq g$. Hence any point of g_{I-B} is a bilateral limit point of g. Hence g is bilaterally dense-in-itself and this completes the proof of Theorem 1.

In proving the above theorem, we obtain the following corollaries similar to theorems in [4].

COROLLARY 1. *A sufficient condition that a bilaterally dense-in-itself function f be connected is that K meets f whenever K is a continuum in the plane such that the X-projection of $K \cap \bar{f}$ is uncountable.*

To see the condition is not necessary, we consider

EXAMPLE 1. Denote by f a function from R into $[0, 1]$ such that f is connected and f is a dense subset of $R \times (0, 1)$. Then the horizontal interval H with X-projection $[0, 1]$ and Y-projection $\{1\}$ has the X-projection of $H \cap \bar{f} = X$-projection of H but misses f.

COROLLARY 2. *A necessary and sufficient condition that a bilaterally dense-in-itself function f be connected is that K meets f whenever K is a continuum in the plane such that the X-projection of $K \cap \bar{f}$ is uncountable with K meeting $f(+)$ and $f(-)$.*

In [3], Ceder has shown that if $\{f_n\}_{n=1}^{\infty}$ is a sequence of real-valued functions with domain the real line such that the union of their graphs has the "intermediate value property", then there exists a function q on the real line with the intermediate value (Darboux) property such that for all x there exists an i such that $q(x) = f_i(x)$. Using Theorem 1 in this paper, we see it is possible to construct a connected function g which agrees with q on a dense subset of the real line such that $\bar{g}$ is the same as the closure of the union of the graphs of the sequence $\{f_n\}_{n=1}^{\infty}$.

We have been unable to improve Theorem 1 so that g is almost continuous. The best we have been able to do is

THEOREM 2. *Let $f : I \to I$ be a connected function such that the Y-projection of $l_x \cap \bar{f}$ is the range of f whenever x is a point of discontinuity of f. Then there exists an almost continuous function q such that $\bar{q} = \bar{f}$.*

In order to prove Theorem 2 we need the notion of a blocking set due to Kellum and Garrett in [5].

DEFINITION 1. The statement that the subset C of I^2 is a *blocking set of f relative to I^2* means that C is closed, C contains no point of f, and C intersects g whenever g is a continuous function from I into I. If no proper subset of C is a blocking set of f relative to I^2, C is said to be a *minimal blocking set of f relative to I^2*.

PROOF OF THEOREM 2. The proof follows the same approach as the proof of Theorem 1 and hence the proof will only be sketched with specifics listed only

when major differences occur. Denote by S the set of discontinuities of f which have the property that each open set containing the discontinuity p contains uncountably many discontinuities of f on either side of p. Denote by T a countable dense subset of I such that the closure of f_T (taken in I^2) is $\bar{f}$. Let $A = (I - S) \cup T$.

Denote by C the family of all closed sets K in I^2 such that the X-projection of $K \cap \bar{f}_S$ is uncountable and K misses f_A. As in Theorem 1 (using the same notation), it follows by an induction argument that there exists a one-to-one function h from C into $\bigcup\{\Psi(K) : K \text{ in } C\}$ such that $h(K)$ is in $\Psi(K)$. We now define a function q exactly as g was defined in Theorem 1.

We claim q is almost continuous, for suppose not. Then by Theorem 1 of [5], there exists a minimal blocking set M for q relative to I^2 such that the X-projection of M is connected. Hence the X-projection of M is a closed, connected, nondegenerate set. Now since a function of Baire class 1 is almost continuous if it is connected (see Theorem 2 of [1]), the X-projection of M must contain a point of S. Therefore the X-projection of $M \cap \bar{f}_S$ is uncountable since (1) the X-projection of M meets S, and (2) the Y-projection of $l_x \cap \bar{f}$ is the range of f whenever x is a point of discontinuity of f. Hence, by construction, M meets q and this is a contradiction. Therefore q is almost continuous.

REMARK. The above construction can easily be used to construct an almost continuous function whose closure is the same as the non-almost continuous (but connected) function exhibited by Jones and Thomas in [5].

In hope of dropping some of the hypotheses in Theorem 2, we proved

LEMMA 1. *Denote by f a connected function from I into I. Then if M is a blocking set of f relative to I^2, then the X-projection of $M \cap \bar{f}$ is infinite.*

PROOF. Denote by f a connected function which is not almost continuous, and denote by M a guaranteed minimal blocking set of f relative to I^2. Suppose that the X-projection of $M \cap \bar{f}$ is finite. Since f is connected, $\bar{f} = \bigcap_{i=1}^{\infty} O_i$, where each O_i is the open set consisting of all points (x,y) such that x and y are in I and the distance from (x,y) to $\bar{f}$ is less than $1/10^i$.

Now each O_i is simply connected and Thomas has shown that if P is a simply connected open set in I^2 which contains a connected function $f : I \to I$ and if F is a finite subset of I, then there is a continuous function g such that $g(x) = f(x)$ for each x in F and g lies in P [12]. Remove a finite number of vertical intervals from each O_i (those intervals corresponding to the finite number of points in the X-projection of $M \cap \bar{f}$) so that one obtains the simply connected open set D_i such that $\bigcap_{i=1}^{\infty} D_i$ contains f, $D_{i+1} \subseteq D$, and $\bigcap_{i=1}^{\infty} D_i$ misses M. Now since the D_i's are nested, there is a positive integer N such that if $i \geqslant N$, D_i misses M. By Thomas' result above, D_N contains a continuous function $g : I \to I$ such that g misses M and this is a contradiction. Hence the X-projection of $M \cap \bar{f}$ is infinite.

QUESTION. Can the hypothesis in Theorem 2 that the Y-projection of $l_x \cap \bar{f}$ is the range of f whenever x is a point of discontinuity of $f : I \to I$ be removed? Note that to do this all one needs to show is that the X-projection of $M \cap \bar{f}$ is dense-in-itself since it is already closed and use the same proof as in Theorem 2.

REMARK. The above question has recently been answered affirmatively by the author and will appear at a later time.

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