INDUCED AUTOMORPHISMS AND SIMPLE APPROXIMATIONS

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ABSTRACT. A class of ergodic, measure preserving invertible point transformations, which are said to admit simple approximations is defined below. If $T$ is an automorphism which admits a simple approximation, conditions are given on a set $A$ so that the induced automorphisms $T^A$ and $T_A$ again admit simple approximations.

1. Preliminaries. Let $(X, F, \mu)$ be a measure space isomorphic to the unit interval with Lebesgue measure. A measure preserving invertible point transformation of $X$ is called an automorphism of $(X, F, \mu)$.

**Definition 1.** A finite ordered collection $\xi = \{A_i: 1 \leq i \leq m\}$ of pairwise disjoint measurable sets in $X$ is called a partition. If the union of members of $\xi$ is $X$, then $\xi$ is called a partition of $X$. If $A \in F$ we write $A \subseteq \xi$ if $A$ is a union of members of $\xi$. If $\eta = \{B_j: 1 \leq j \leq n\}$ is a partition, we write $\eta \subseteq \xi$ if $B_j \subseteq \xi$ for $j = 1, \ldots, n$.

**Definition 2.** Let $e$ denote the partition of $X$ into single points. We shall say that a sequence of partitions $\{\xi(n)\}$ converges to the unit partition, and we write $\xi(n) \to e$ if for each $A \in F$, $\mu(A \Delta A(\xi(n))) \to 0$ as $n \to \infty$, where $A(\xi(n)) \subseteq \xi(n)$ and is such that $\mu(A \Delta A(\xi(n)))$ is a minimum.

Following, we define the class of automorphisms that admit simple approximation.

**Definition 3.** An automorphism $T$ is said to admit a simple approximation if there exists a sequence of partitions $\{\xi(n)\}$ with the property that

(i) $\xi(n) \to e$ as $n \to \infty$,
(ii) $TC_i(n) = C_{i+1}(n)$ for $i = 1, \ldots, q(n) - 1$.

Chacon and Schwartzbauer [2] require the additional condition that $\lim_{n \to \infty} q(n)\mu(X \backslash \bigcup_{i=1}^{q(n)} C_i(n)) = 0$. This condition will not be required by us, but we shall see that a similar condition arises naturally in the discussion of the induced automorphisms $T^A$ and $T_A$. In fact, Schwartzbauer [5] has shown that if $\lim_{n \to \infty} q(n)\mu(X \backslash \bigcup_{i=1}^{q(n)} C_i(n)) = c < \infty$, then $T$ cannot be strongly mixing.

It is well known that automorphisms that admit simple approximation are ergodic and have simple spectrum [5], [2].

2. The induced automorphisms $T^A$ and $T_A$. Let $T: X \to X$ be an automorphism and $A \in F$ a set with positive measure.

**Definition 4.** Let $A'$ be a copy of $A$, $\tau: A \to A'$ a one-to-one map, and $X^A = X \cup A'$. Then the primitive transformation $T^A: X^A \to X^A$ is defined by...
Definition 5. We define the derivative transformation $T_A : A \to A$ by

$$T_A(x) = T^n(x), \quad x \in A,$$

where $n$ is the least integer such that $T^n(x) \in A$ (neglecting sets of measure zero).

Both $T^A$ and $T_A$ are called induced transformations. When $X^A$ and $A$ are made into probability spaces in the obvious way, then $T^A$ and $T_A$ become automorphisms which are ergodic if and only if $T$ is ergodic.

Kakutani [3] first introduced the idea of induced transformation, and in [4] he gave an example of an induced automorphism which is weakly mixing but not strongly mixing. In this example the underlying automorphism $T$ admits a simple approximation.

Definition 6. We can define a metric $\rho$ on the set of ordered partitions with $m$ elements (neglecting sets of measure zero).

If $\xi = \{A_i : i = 1, \ldots, m\}, \eta = \{B_j : j = 1, \ldots, m\}$ put

$$\rho(\xi, \eta) = \sum_{i=1}^{m} \mu(A_i \triangle B_i).$$

The measure algebra $(F, \mu)$ is a complete metric space with respect to the metric $d$ given by

$$d(A, B) = \mu(A \triangle B), \quad A, B \in F.$$

We need shall the following lemma (Baxter [1]).

Lemma 1. Let $\xi(n) = \{A_i(n) : i = 1, \ldots, q(n)\}, \eta(n) = \{B_j(n) : j = 1, \ldots, q(n)\}$ be sequences of partitions such that $\xi(n) \to \xi$ and $\rho(\xi(n), \eta(n)) \to 0$, then $\eta(n) \to \xi$.

3. Main theorems. We shall prove the results for the primitive automorphism $T^A$ and will outline the proofs for the derived automorphism $T_A$.

Theorem 1. Let $T : X \to X$ be an automorphism which admits a simple approximation; then there is a set of subsets of $X$, dense in $F$, such that the induced automorphisms $T^A$ and $T_A$ on any one of these sets also admit a simple approximation.

Proof. By a result of Baxter [1] we may assume that $T$ admits a simple approximation with respect to an increasing sequence of partitions $\xi(n)$, i.e. $\xi(n) \preceq \xi(n + 1)$ for all $n$.

Let $\xi(n) = \{C_i(n) : i = 1, \ldots, q(n)\}$ and fix $m \geq 1$. If we put $A = C_j(m)$ for some $j$, $1 \leq j \leq q(m)$, then it is easy to see that $T^A$ again admits a simple approximation, and so the result follows. The proof for $T_A$ is similar.

Lemma 2. Let $T : X \to X$ be an automorphism. If $A$ is a measurable set with
positive measure which can be approximated by a sequence of measurable sets
\( A(n) \subset A \) in the sense that \( \mu(A \setminus A(n)) \to 0 \) as \( n \to \infty \), then the sequence of
transformations \( \{T_n\} \) defined by

\[
T_n(x) = \begin{cases} 
T^{A(n)}(x), & x \in A'(n) \cup X, \\
x, & x \in A \setminus A'(n),
\end{cases}
\]

converges to \( T^A \) in the uniform topology. \( (T^{A(n)}) \) is the primitive automorphism
induced by \( T \) on \( A(n) \). A'(n) and A' are copies of A(n) and A respectively.

PROOF. Clearly the automorphisms \( T^A \) and \( T_n \) coincide on the sets \( A(n), X \setminus A \) and \( A'(n) \), so they can only differ on the sets \( A \setminus A(n), A' \setminus A'(n) \). Therefore

\[
\mu\{x : T_n(x) \neq T^A(x)\} \leq \mu([A \setminus A(n)] \cup [A' \setminus A'(n)])
\leq 2\mu(A \setminus A(n)) \to 0 \quad \text{as} \; n \to \infty.
\]

Hence \( T_n \to T^A \) in the uniform topology.

REMARK. The corresponding result for the derived automorphism \( T_\delta \) is true
provided we assume, in addition, that the automorphism \( T \) admits a simple
approximation and that \( A(n) \leq \xi(n) \).

Following is our main theorem.

THEOREM 2. Let \( T \) admit a simple approximation with respect to a sequence of
partitions \( \{\xi(n)\} \), \( \xi(n) \) having \( q(n) \) elements, and suppose \( A \in F \) with \( \mu(A) > 0 \)
can be approximated by sets \( A(n) \subset A \) with \( A(n) \leq \xi(n) \) and such that
\( q(n)\mu(A \setminus A(n)) \to 0 \) as \( n \to \infty \). Then \( T^A \) and \( T_\delta \), the induced automorphisms on
\( A \), admit a simple approximation.

REMARK. We prove the theorem for the primitive automorphism \( T^A \). The
proof for the derived automorphism \( T_\delta \) is similar.

PROOF. \( A(n) \leq \xi(n) \), so assume that \( A(n) \) is the union of \( p(n) \) elements of
\( \xi(n), n = 1, 2, \ldots A'(n) \subset A' \), so we can construct a sequence of partitions
for \( X \cup A' \) consisting of the \( q(n) \) elements of \( \xi(n) \) together with the \( p(n) \)
elements of \( A'(n) \) (which are just copies of the \( \xi(n) \)-sets of \( A(n) \)). Denote this
partition by \( \beta(n) \) and give it the natural order obtained from the transformation
\( T^{A(n)} \).

Put \( \beta(n) = \{D_i(n) : i = 1, \ldots, p(n) + q(n)\} \). Clearly, as \( n \to \infty \beta(n) \to \varepsilon^A, \)
the point partition of \( X \cup A' \), and also

\[
D_i(n) = T_n^{-1} D_1(n) \quad \text{for} \; i = 1, \ldots, p(n) + q(n),
\]

where \( T_n \) is the automorphism defined in Lemma 2.

Define a second sequence of partitions for \( X \cup A' \), denoted by \( \{\eta(n)\} \) where

\[
\eta(n) = \{E_i(n) : i = 1, \ldots, p(n) + q(n)\}
\]

and

\[
E_i(n) = D_i(n), \quad E_1(n) = (T_\delta)^{-1} D_1(n), \quad i = 1, \ldots, p(n) + q(n).
\]
We shall show that $T^A$ admits a simple approximation with respect to $\eta(n)$. It suffices to show that $\eta(n) \to \varepsilon^A$ as $n \to \infty$. We show that $\rho(\beta(n), \eta(n)) \to 0$, and since $\beta(n) \to \varepsilon^A$, the result will follow from Lemma 1.

$$\rho(\beta(n), \eta(n)) = \sum_{i=1}^{p(n)+q(n)} \mu(D_i(n) \triangle E_i(n))$$

$$= \sum_{i=0}^{p(n)+q(n)-1} \mu(T^i\text{-}D_1(n) \triangle (T^A)^i\text{-}D_1(n)).$$

But $T_n$ approximates $T^A$ in the uniform topology. In fact if $x \in D_1(n)$ then

$$T^i_n(x) = (T^A)^i(x)$$

unless

$$x \in \bigcup_{i=0}^{i-1} (T^A)^{-i}[(A\setminus A(n)) \cup (A^*\setminus A^*(n))],$$

i.e. unless $x \in \bigcup_{i=0}^{i-1} (T^A)^i(G(n))$ where $G(n) = (A\setminus A(n)) \cup (A^*\setminus A^*(n))$. It follows that

$$\frac{1}{2} \mu(T^i\text{-}D_1(n) \triangle (T^A)^i\text{-}D_1(n)) \leq \mu\left[ \left( \bigcup_{i=0}^{i-1} (T^A)^{-i}G(n) \right) \cap D_1(n) \right]$$

$$\leq \mu\left[ \sum_{i=0}^{p(n)+q(n)-1} (T^A)^{-i}G(n) \cap D_1(n) \right]$$

$$\leq \frac{p(n)+q(n)-1}{2} \mu[(T^A)^{-i}G(n) \cap D_1(n)]$$

$$= \sum_{i=0}^{p(n)+q(n)-1} \mu[G(n) \cap (T^A)^i\text{-}D_1(n)]$$

$$= \mu[G(n) \cap \bigcup_{i=0}^{p(n)+q(n)-1} (T^A)^i\text{-}D_1(n)]$$

$$\leq \mu(G(n)) \leq 2\mu(A\setminus A(n)).$$

Hence

$$\frac{1}{2} \rho(\beta(n), \eta(n)) \leq 2 \sum_{i=0}^{p(n)+q(n)-1} \mu(A\setminus A(n))$$

$$= 2(p(n) + q(n))\mu(A\setminus A(n))$$

$$\leq 4q(n)\mu(A\setminus A(n))$$

$$\to 0 \quad \text{as} \quad n \to \infty.$$
\( q(n)\mu(X \setminus \bigcup_{i=1}^{p(n)} C_i(n)) \to c < \infty \), then \( T \) is not strongly mixing. From this we deduce

**Corollary 1.** Suppose that \( T \) and \( A \in F \) satisfy the hypothesis of Theorem 2; then the derived automorphism \( T_A : A \to A \) is not strongly mixing.

**Remark.** We cannot deduce that the primitive automorphism \( T^A \) is also not strongly mixing.

**Bibliography**


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