THE STARLIKE RADIUS FOR CLASSES OF REGULAR BOUNDED FUNCTIONS[1]

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ABSTRACT. Let $B_0(a)$ be the class of all functions f defined on $|z| < 1$ such that (i) $f(z)$ is regular, (ii) $|f(z)| < 1$, (iii) $f(0) = 0$, (iv) $0 <|f'(0)| = a \leqslant 1$. For fixed R, $a \leqslant R < 1$, let $B_0(a; R)$ be that subclass having nonzero zeros at $z = z_k$, $k = 1, 2, \ldots$, such that $\prod|z_k| = R$. The subclass having no nonzero zeros is designated as $B_0(a; 1)$. A sharp lower bound for $\mathrm{Re}[zf'(z)/f(z)]$ for the class $B_0(a; R)$, $a \leqslant R \leqslant 1$, is obtained, and the radius of starlikeness is found. A covering theorem for the class is also obtained.

Introduction. Dieudonné [1, pp. 349–352] obtained the starlike radius for the class $B_0(a)$ of functions f which are regular and bounded on the unit disc and which are normalized by (i) $f(0) = 0$, (ii) $f'(0) = a > 0$ and (iii) $|f(z)| \leqslant 1$. Here we will find the starlike radius for a subclass of $B_0(a)$. Suppose $f(z) \in B_0(a)$ has zeros at $z = z_k$, $k = 1,2, \ldots$, where $0 <|z_k| < 1$ for all k. For fixed R, $a \leqslant R < 1$, designate as $B_0(a; R)$ the subclass of functions f such that $\prod_k|z_k| = R$ and $f(z_k) = 0$. For completeness the subclass having no nonzero zeros for $|z| < 1$ is designated as $B_0(a; 1)$. The starlike radius will be obtained by finding a sharp lower bound for $\mathrm{Re}[zf'(z)/f(z)]$ on $|z| = r < 1$. A covering theorem will also be given.

The following two representation theorems form the basis of the subsequent analysis.

THEOREM. *Let* $f \in B_0(a; 1)$. *Then, for* $|z| < 1$, *there is defined on* $[-\pi, \pi]$ *a nondecreasing function,* $\alpha(t)$, *with* $\alpha(\pi) - \alpha(-\pi) = 1$, *such that*

$$
(1) \qquad f(z) = z \exp\left\{ (\log a) \int_{-\pi}^{\pi} (1 + e^{it}z)(1 - e^{it}z)^{-1} \, d\alpha(t) \right\}.
$$

THEOREM. *Let* $f \in B_0(a)$ *and let* $\{z_k\}$, *counted according to their multiplicity, be the nonzero values where* $f(z) = 0$. *Then* f *is uniquely expressible in the form*

$$
(2) \qquad f(z) = h(z)\prod\left(\frac{\bar{z}_k}{|z_k|} \cdot \frac{z_k - z}{1 - \bar{z}_kz} \right), \qquad |z| < 1,
$$

where $h(z) \in B_0(a/R; 1)$ *and* $R = \prod|z_k|$.

The first theorem is proved by applying Herglotz's Theorem (cf. [6, p. 275]) to

$$F(z) = \begin{cases} \log[f(z)/z], & z \neq 0, \\ \log a, & z = 0. \end{cases}$$

The second theorem is an adaptation of the Blaschke representation theorem(cf. [4, pp. 63–67]).

Preliminary relations. The following relations, which are used in the derivation, are interesting in themselves.

LEMMA. *Let* $\{t_k\}$, $0 < t_k < 1$, *be given such that* $\prod t_k = T > 0$. *Then*

$$(3) \qquad\qquad 0 \leqslant 1/T - T - \sum(1/t_k - t_k).$$

Equality occurs only if $\{t_k\} = \{T\}$. *This result is the best possible in the sense that for any* T, $0 < T < 1$, *there exists an admissible sequence* $\{t_k\}$, *either finite (with any specified nonzero number of elements) or infinite, such that the right side of* (3) *is arbitrarily small.*

The inequality can be proved by mathematical induction for the finite case and by a limit process for the infinite case. That the result is "best possible" can be demonstrated by noting that, for any sequence $\{t_k\}$ having $t_1 = T + \theta(1 - T)$, $0 < \theta < 1$, and $\prod t_k = T$,

$$(4) \qquad \frac{1}{T} - T - \sum\left(\frac{1}{t_k} - t_k\right) < \frac{1}{T} - T - \left(\frac{1}{t_1} - t_1\right) < \frac{\theta}{T^2}.$$

Inequality (3) is equivalent to the following

THEOREM. *Let* $\{r_k\}$, $0 < r_k < 1$, *with* $\prod r_k = R > 0$ *be given. For each* r, $0 \leqslant r < R$,

$$(5) \qquad 0 \leqslant \frac{1 - R^2}{(R - r)(1 - Rr)} - \sum \frac{1 - r_k^2}{(r_k - r)(1 - r_k r)}.$$

Equality occurs only if $\{r_k\} = \{R\}$. *This result is the best possible in the sense of the previous Lemma.*

We first expand the right side of (5) in a Taylor series. Thus, we want to show

$$0 \leqslant \sum_j r_j\left[\frac{1}{R^{j+1}} - R^{j+1} - \sum_k\left(\frac{1}{r_k^{j+1}} - r_k^{j+1}\right)\right].$$

We associate r_k^{j+1} and R^{j+1} with t_k and T of (3) to complete the proof. A result similar to (4) shows (5) is best possible.

(5) can be integrated with respect to r to give

$$(6) \qquad \prod (r_k - r)(1 - r_k r)^{-1} \geqslant (R - r)(1 - rR)^{-1}, \qquad 0 \leqslant r < R.$$

The starlike radius. The starlike radius ρ for a class of functions is the least upper bound of radii r of discs $|z| \leqslant r$ that are mapped by an arbitrary function $f(z)$ of the class onto a domain having the property that $\arg f(z)$

always increases as the point $f(z)$ moves around the boundary in the positive direction. Since $\partial [\arg f(z)]/\partial\theta$ equals $\mathrm{Re}[zf'(z)/f(z)]$, where $z = r \exp(i\theta)$, we must find the values r for which $\mathrm{Re}[zf'(z)/f(z)] \geqslant 0$ on $|z| \leqslant r$ (cf. [2, p. 167]).

It can be shown, by means of (1), that if $f \in B_0(a; 1)$ then, for $|z| \leqslant r < 1$,

$$\mathrm{Re}\big[zf'(z)/f(z)\big] \geqslant 1 + 2r(1 - r)^{-2}(\log a).$$

That this result is sharp can be demonstrated by the function

(7) $$f(z) = z \exp\big\{(\log a)(1 + z)(1 - z)^{-1}\big\}.$$

This knowledge of $B_0(a; 1)$ can be used with the Blaschke product to give the main result of this section.

THEOREM. *Let $f \in B_0(a; R)$. Then, for $|z| \leqslant r < R \leqslant 1$,*

(8) $$\mathrm{Re}\big[zf'(z)/f(z)\big] \geqslant 1 + \frac{2r}{(1 - r)^2} \log \frac{a}{R} - \frac{r(1 - R^2)}{(R - r)(1 - Rr)} .$$

This result is sharp.

For a given set $\{z_k\}$, $0 < |z_k| = r_k < 1$, such that $\Pi r_k = R > 0$, let

$$g(z) = \prod\left(\frac{\bar{z}_k}{|z_k|} \cdot \frac{z_k - z}{1 - \bar{z}_k z} \right).$$

Then, for $r < g \cdot 1 \cdot b \cdot r_k$, on $|z| = r$, we can show

(9) $$\mathrm{Re}\left[\frac{zg'(z)}{g(z)} \right] \geqslant -\sum \frac{r(1 - r_k^2)}{(r_k - r)(1 - r_k r)} .$$

Equality occurs if and only if $z = re^{i\alpha}$ and $z_k = r_k e^{i\alpha}$ for all k and some real α. We can combine (2), (5), (7), and (9) to obtain the result stated.

The radius of starlikeness ρ for the class is the smallest positive zero of the right side of inequality (8). If $R = 1$, this value can be obtained explicitly:

$$\rho\big[B_0(a; 1)\big] = 1 - \log a - \big[(1 - \log a)^2 - 1\big]^{\frac{1}{2}}$$

[7, pp. 1031–1032].

To consider the class $B_0(a)$, note for $a \leqslant R \leqslant 1$,

$$1 + \frac{2r}{(1 - r)^2} \log a \geqslant 1 + \frac{2r}{(1 - r)^2} \log \frac{a}{R} - \frac{r(1 - R^2)}{(R - r)(1 - Rr)}$$

$$\geqslant 1 - \frac{r(1 - a^2)}{(a - r)(1 - ar)}$$

THEOREM. *Let $f \in B_0(a)$. Then, for $|z| \leqslant r < a$,*

$$\mathrm{Re}\big[zf'(z)/f(z)\big] \geqslant 1 - r(1 - a^2)\big[(a - r)(1 - ar)\big]^{-1}.$$

This result is sharp.

From this we note that

$$\rho\big[\,B_0(a)\,\big] = \Big[\,1 - (1 - a^2)^{\frac{1}{2}}\,\Big]/a$$

[1, pp. 349–352].

The following theorem shows the monotonicity of the starlike radius.

THEOREM. *Let* a, R_1, R_2, $0 < a \leqslant R_1 < R_2 \leqslant 1$, *be given. Then*

$$\rho\big[\,B_0(a)\,\big] \leqslant \rho\big[\,B_0(a;\,R_1)\,\big] < \rho\big[\,B_0(a;\,R_2)\,\big].$$

COVERING THEOREM. If D is a point set in the complex plane, $f(z)$ is said to cover D univalently in K if f is univalent in a subregion K_1 of K and $f(K_1)$ covers D [5, p. 374].

If $f \in B_0(a;\,1)$, then, for $|z| = r < 1$, as a direct consequence of (1), we obtain

$$r \exp\Big\{(\log a)\,\frac{1 + r}{1 - r}\Big\} \leqslant |f(z)| \leqslant r \exp\Big\{(\log a)\,\frac{1 - r}{1 + r}\Big\}.$$

When this inequality is used in conjunction with the Blaschke product representation and inequality (6), the following results:

THEOREM. *If* $f \in B_0(a;\,R)$, $a \leqslant R \leqslant 1$, *then the map of* $|z| < 1$ *by* $w = f(z)$ *covers univalently the disc*

$$|w| < \rho(R - \rho)(1 - R\rho)^{-1} \exp\big\{(\log a/R)(1 + \rho)(1 - \rho)^{-1}\big\},$$

where $\rho = \rho[\,B_0(a;\,R)\,]$. *This result is sharp in that no larger disc, centered at the origin, is univalently covered by all* f *of the class.*

The function demonstrating sharpness is

$$f(z) = z\,\frac{R - z}{1 - Rz}\,\exp\Big\{(\log a)\,\frac{1 + z}{1 - z}\Big\}.$$

These results show that the radius of starlikeness for the class is also the radius of univalence.

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