IDEMPOTENT MAXIMAL IDEALS AND INDEPENDENT SETS

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Abstract. Let $E$ be a compact independent subset of a nondiscrete LCA group $G$. Let $G_pE$ be the subgroup of $G$ generated algebraically by $E$. If $\mu$ is a continuous, regular, Borel measure on $G_pE$ with $\mu(G_pE) \neq 0$, then there exists a maximal ideal $x$ of the algebra $M(G)$ of regular Borel measures on $G$ such that the restriction of $x$ to $L^1(\mu) = \{v \in M(G) : v \ll \mu\}$ is a nontrivial idempotent in $L^\infty(\mu)$. This result is used to give a new proof that $G_pE$ has zero Haar measure.

Introduction. A subset $E \subseteq G$ of a LCA group $G$ is independent if $n \geq 1$, $x_1 \in G, \ldots, x_n \in G$, $m_1 \in \mathbb{Z}, \ldots, m_n \in \mathbb{Z}$ and

$$\sum_{j=1}^{n} m_j x_j = 0 \implies m_1 x_1 = \cdots = m_n x_n = 0.$$

Every maximal ideal (multiplicative linear functional) $x$ of $M(G)$ induces an element $x_M$ of $P(J)$, $v \in M(G)$, through restriction of $x$ to $L^1(x) = \{v : v < x\}$, ($v < x$ means $v$ is absolutely continuous with respect to $x$.) An idempotent $f \in L^\infty(\mu)$ is a nontrivial idempotent if $f'(0)$ and $f'(1)$ both have nonzero $\mu$-measure. For more about maximal ideals, see [GRS], [Sr], [T].

We prove:

Theorem 1. Let $E$ be a compact independent subset of a LCA group $G$. Suppose $\mu$ is a continuous measure on $G$ with $\mu(G_pE) \neq 0$. Then there exists a maximal ideal $x$ of $M(G)$ such that $x_M$ is a nontrivial idempotent in $L^\infty(\mu)$.

Corollary 1. If $H$ is a $\sigma$-compact, nondiscrete, subgroup of $G$, and $E \subseteq G$ is compact, and independent, then $(G + G_pE) \cap H$ has zero $H$-Haar measure, for all $x \in G$.

Corollary 2. Let $G$ be compact, and $\mu$ a Riesz product on $G$ with $\limsup |\hat{\mu}(\gamma)| < 1$. If $E \subseteq G$ is compact and independent then $\mu(x + G_pE) = 0$ for all $x \in G$.

Corollary 3. Let $\mu$ be any continuous measure on $G$ such that $|\hat{x}|$ is constant a.e. $d\mu$ for all maximal ideals $x$ of $M(G)$. Then $\mu(G_pE) = 0$ whenever $E$ is an independent set.

Remarks. (i) Corollary 1 has a long history. It was first stated in a weaker form by Rudin in [Ru, 5.3.6], but the proof given there is obscure. A clearer...
proof appeared in our previous note [G]. Later, in his 1972 Northwestern University Thesis, Rago proved Corollary 1 as a corollary of another result, most simply stated this way: if \( \mu_1, \ldots, \mu_{n+1} \) are continuous measures on \( G \) and \( E \) is independent, then \( \mu_1 \ast \cdots \ast \mu_{n+1}(n(E \cup \neg E)) = 0 \). Salinger and Varopoulos [SV, Theorem 1] had proved this for \( n = 1 \) and metrizable \( G \). Rago’s result appears in his [Ra].

(ii) This paper grew out of an attempt to learn more about sets on which Riesz products may be concentrated. Corollary 2 is the best we have obtained so far. It seems possible that, for a Riesz product on the circle \( \mu \), and a proper Borel subgroup \( H \) of the circle, \( \mu(H) = 0 \). Corollary 2 is a trivial consequence of Theorem 1 and the deep work of Gavin Brown.

(iii) The conclusion of Corollary 3 holds for “most” infinite convolutions \( \mu \) of discrete probability measures: \( |\chi| \) is constant a.e. \( d\mu \).

(iv) Finally, the idea of the proof of Theorem 1 is this. If \( \mu(GpE) \neq 0 \), then \( \mu(n(E \cup \neg E)) \neq 0 \) for some \( n \geq 1 \). The proof of Theorem 1 is an induction on \( n \). The proof is simple when \( n = 1 \): we may find disjoint compact subsets \( E_1, E_2 \) of \( E \cup \neg E \) with \( \mu(E_1) \neq 0 \neq \mu(E_2) \). We let \( \mathcal{R} \) be the Raikov system [GRS] generated by \( E_1 \) and observe that if \( \pi \) is the projection of \( M(G) \) onto the \( L \)-algebra of measures concentrated on sets in \( \mathcal{R} \), then \( \chi(v) = \int d(\pi v) \) has the required properties. The proof of Theorem 1 (for \( n > 1 \)) is a more complicated version of this observation.

(v) The original version of this paper contained a larger and more cumbersome proof of Theorem 1. We express our thanks to the referee, who gave a simplified version, and for his (her) other helpful comments.

1. Proof of Theorem 1. We have several steps, in all of which we retain the hypotheses of Theorem 1. First observe that we may assume \( \mu \geq 0 \).

(A) Suppose \( \chi_\nu \) is idempotent for all \( \nu \in M(G) \). Then \( \chi_\mu \) is a nontrivial idempotent iff \( \chi_{\delta_x,\ast \mu} \) is a nontrivial idempotent, for all \( x \in G \). [Indeed, \( \chi(\delta_x) = 1 \) so \( \chi_{\delta_x,\ast \mu}(x + y) = \chi_{\delta_x}(x)\chi_{\mu}(y) \) a.e. \( d\mu \).] The assertion now follows.

(B) Since the integers are well ordered and \( \mu \) is continuous, there exists a minimal integer \( m > 0 \) such that (setting \( Q = E \cup \neg E \)):

\[
\begin{align*}
&\exists x \in G \text{ with } \mu(x + mQ) \neq 0 \\
&\forall y \in G \text{ and } 0 \leq j < m, \mu(y + jQ) = 0.
\end{align*}
\]

By using (A) we see we may assume that both

\[
\mu(mQ) \neq 0
\]

and (2) hold. We leave \( m \) fixed. We may further assume (by replacing \( \mu \) with a measure absolutely continuous with respect to \( \mu \)) that \( \mu \) has support \( mQ \).

(C) Let \( x_1, \ldots, x_m \in E \), with \( \pm x_1 \pm \cdots \pm x_m = x \) in the support of \( \mu \), for an appropriate choice of signs. Let \( X = \{x_1, \ldots, x_n\} \). Then

\[
mQ = \bigcup_{j=0}^{m} [j((E \setminus X) \cup -(E \setminus X)) + (m - j)(X \cup \neg X)].
\]

From (2) and (4), we see that...
\( \mu(mQ) = \mu(m(E \setminus \{x_1, \ldots, x_m\}) \cup -(E \setminus \{x_1, \ldots, x_m\})) \).

The regularity of \( \mu \) (and continuity of addition in \( G \)) imply

\[ \mu(mQ) = \sup \mu(m(E \setminus W) \cup -(E \setminus W)) \]

where the supremum in (6) is taken over (small) neighborhoods \( W \) of \( \{x_1, \ldots, x_m\} \). But, if \( W \) is any open neighborhood of \( \{x_1, \ldots, x_m\} \), then \( m((E \setminus W) \cup -(E \setminus W)) \) misses an entire (relative) neighborhood of \( \pm x_1 \pm \cdots \pm x_m = x \) in the support of \( \mu \). Therefore

\[ \mu(mQ) > \mu(m(E \setminus W) \cup -(E \setminus W))). \]

Of course, for a sufficiently small neighborhood \( W \) of \( \{x_1, \ldots, x_m\} \), (6) (combined with (7)) yields

\[ \mu(mQ) > \mu(m((E \setminus W) \cup -(E \setminus W))) > \frac{1}{2} \mu(mQ) > 0. \]

(D) We do some computations with \( mQ \). Assume \( W \) (a neighborhood of \( \{x_1, \ldots, x_m\} \)) has been chosen so (8) holds.

For a set \( F \subseteq G \), and \( j \geq 0 \), set \( (j) \ast F = j(F \cup -F) \), and \( 0 \ast F = \{0\} \). Then

\[ mQ = \bigcup_{j=0}^{m} (j) \ast (E \setminus W) + (m-j) \ast E \cap W \]

and (8) and (9) together imply that for some \( 0 \leq j_1 < m \),

\[ \mu((j_1) \ast (E \setminus W) + (m-j_1) \ast (E \cap W)) > 0. \]

(E) Let \( \mathcal{R} \) be the Raikov system in \( G \) which is generated by \( F \cup -F \) where \( F = E \setminus W \). Let \( \pi \) be the map from \( M(G) \) to \( A_{\mathcal{R}} \), the algebra of measures \( \omega \) on \( G \) such that

\[ \|\omega\| = \sup \{\|\omega\|(y_1 + GpF \cup \cdots \cup y_j + GpF): y_1, \ldots, y_j \in G, 1 \leq j < \infty\}. \]

(Facts about \( A_{\mathcal{R}} \) and \( \pi \) can be found in [GRS].)

Then the definition of \( \pi \), (8), and (11) together imply

\[ \|\pi\| \geq \mu((m) \ast F) = \mu(m((E \setminus W) - (E \setminus W))) > 0. \]

Let \( \chi \) be the maximal ideal of \( M(G) \) defined by

\[ \chi(\omega) = (\pi\omega)^*(0); \quad \omega \in M(G). \]

Straightforward computations show that

\[ \begin{cases} x_{\pi\omega} = 1 & \text{a.e. } d\pi\omega, \\ x_{(\omega-\pi\omega)} = 0 & \text{a.e. } d(\omega - \pi\omega), \end{cases} \]

so \( x_\omega \) is an idempotent for all \( \omega \in M(G) \). In particular \( \chi(\mu) = \|\pi\| > 0 \), so
formula (12), applied to $\mu$ shows (using (11)) that $\chi_\mu = 1$ on a set of nonzero $\mu$-measure. We must show that $\chi_\mu = 0$ on another set of nonzero $\mu$-measure. This will complete the proof that $\chi_\mu$ is a nontrivial idempotent.

(F) Set $K = (j_1) \ast F + (r) \ast L$, where $r = m - j_1$, and $L = E \cap W$. We wish to prove

$$(13) \quad \mu(K \cap (y + GpF)) = 0 \quad \text{for all } y \in G.$$ 

Then (11) and (13) imply $\pi_\mu(K) = 0$ while (10) implies $\mu(K) > 0$. This shows that $\chi_\mu(x) \equiv 0$ a.e. $d\mu$ for $x \in K$.

So suppose $K \cap (y + GpF) \neq \emptyset$, for some $y \in G$. Then (from the definition of $K$), there exist $x \in (r) \ast L$, and $a \in GpF$ such that $y = x + a$. If $z \in K \cap (y + GpF)$, then for some $b \in GpF$, $c \in (j_1) \ast F$, $d \in (r) \ast L$,

$$z = c + d = y + b = (a + b) + x.$$ 

Since $E$ is independent, $(GpF) \cap (GpL) = \emptyset$, so $c = a + b$ and $d = x$. Therefore

$$z = x + c \in x + (j_1) \ast F.$$ 

Therefore

$$K \cap (y + GpF) \subseteq x + (j_1) \ast F,$$

which has $\mu$-measure zero, by (2).

2. Proof of corollaries.

Proof of Corollary 1. If $\lambda$ is $H$-Haar measure restricted to any set of finite $H$-Haar ($\lambda$) measure, and $\chi$ is any maximal ideal of $M(G)$, then $\chi_\lambda$ may not be a nontrivial idempotent, since $\chi_\lambda^*$ on $L^1(\lambda) = \{v \in M(G): v \text{ is absolutely continuous with respect to } \lambda \}$, is either zero, or agrees $\lambda$-almost everywhere with a (unimodular) character on $H$. This proves the corollary when $x = 0$. Now apply paragraph (A) of the proof of Theorem 1.

Proof of Corollary 2. Brown [B] shows each $\chi$ has $|\chi_\mu|$ constant a.e. $d\mu$ if $\mu$ is a Riesz product with $\lim \sup |\hat{\mu}(\gamma)| < 1$.

References


[Originally Transl. 81 (1953).] MR 12, 420; 14, 768.


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