EXTENSIONS OF CONTINUOUS FUNCTIONS
FROM DENSE SUBSPACES

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Abstract. Let $X$ and $Y$ be topological spaces, let $S$ be a dense subspace of $X$, and let $f: S \to Y$ be continuous. When $Y$ is the real line $\mathbb{R}$, the Lebesgue sets of $f$ are used to provide necessary and sufficient conditions in order that the (bounded) function $f$ have a continuous extension over $X$. These conditions yield the theorem of Taimanov (resp. of Engelking and of Blefko and Mrówka) which characterizes extendibility of $f$ for $Y$ compact (resp. realcompact). In addition, an extension theorem of Blefko and Mrówka is sharpened for the case in which $X$ is first countable and $Y$ is a closed subspace of $\mathbb{R}$.

We first quote (in Theorem 1) two basic results concerning extension of a continuous function from a dense subspace of a topological space. Theorem 1A is due to Taimanov [10] (see also [5, Theorem 3.2.1]) and, in dual form, to Eilenberg and Steenrod [3, Lemma 10.9.6] (cf. [5, Exercise 3.2A]). Theorem 1B is due, independently, to Engelking [4, Theorem 2] and to Blefko and Mrówka [2, Theorem A]. (Theorem A of [2] includes the unneeded hypothesis that $X$ is $T_1$.) For additional results on extension of continuous functions from dense subspaces, see McDowell [7].

Theorem 1. Let $X$ and $Y$ be topological spaces, let $S$ be a dense subspace of $X$, and let $f: S \to Y$ be continuous.

A (TAIMANOV). If $Y$ is compact (Hausdorff), then these are equivalent:
1. $f$ extends continuously over $X$.
2. If $F_1$ and $F_2$ are disjoint closed subsets (or, alternatively, zero-sets) of $Y$, then $f^{-1}(F_1)$ and $f^{-1}(F_2)$ have disjoint closures in $X$.

B (ENGELKING AND BLEFKO-MRÓWKA). If $Y$ is realcompact, then these are equivalent:
1. $f$ extends continuously over $X$.
2. If $\{F_n\}_{n=1}^\infty$ is any sequence of closed subsets (or, alternatively, zero-sets) of $Y$ with $\bigcap_{n=1}^\infty F_n = \emptyset$, then $\bigcap_{n=1}^\infty \text{cl}_X f^{-1}(F_n) = \emptyset$.

By a zero-set is meant the set of zeros of a real-valued continuous function. For the theory of realcompact spaces, see Gillman and Jerison [6].

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In this note we obtain a sharper version of Theorem 1A (resp. 1B) for the special case in which \( f \) is a bounded continuous function (resp. continuous function) from \( S \) into the real line \( \mathbb{R} \); this is Theorem 2 below. Theorem 1, in turn, will follow readily from Theorem 2. We also include a sharpening (for real-valued functions) of a theorem of Blefko and Mrówka concerning extension of a continuous function from a dense subspace of a first countable space [2, Theorem D] (see Theorem 3 below).

If \( X \) is a topological space, then \( C(X) \) (resp. \( C^*(X) \)) will denote the set of all continuous (resp. bounded continuous) real-valued functions on \( X \). If \( f \in C(X) \) and \( a \in \mathbb{R} \), we set

\[
L_a(f) = \{ x \in X : f(x) \leq a \}, \quad L^a(f) = \{ x \in X : f(x) \geq a \}.
\]

Sets of the form \( L_a(f) \) or \( L^a(f) \) are Lebesgue sets of \( f \). The point of Theorem 2 (which may be viewed as an analogue of [6, 1.18]) is that it characterizes extendibility of \( f \) in terms of the Lebesgue sets of \( f \). (Theorem 2 is thus a fragment of a general program whereby real-valued functions are studied by means of their Lebesgue sets; see, e.g., [8], [9], and [1, §§2–3]. Other aspects of this program will be treated by the author elsewhere.)

**Theorem 2.** Let \( S \) be a dense subspace of a topological space \( X \), let \( f \in C(S) \), and consider these conditions on \( f \):

(a) \( f \) extends continuously over \( X \).

(b) Disjoint Lebesgue sets of \( f \) have disjoint closures in \( X \).

(c) \( \bigcap_{n=1}^{\infty} \text{cl}_X (L_{-n}(f) \cup L^n(f)) = \emptyset \).

Then (a) is equivalent to the conjunction of (b) and (c); and if \( f \in C^*(S) \), (a) is equivalent to (b).

**Proof.** First assume (a), so that \( f = g|S \) for some \( g \in C(X) \). To verify (b), we need only note that if \( a < b \), then

\[
\text{cl}_X L_a(f) \cap \text{cl}_X L_b(f) \subset L_a(g) \cap L_b(g) = \emptyset.
\]

To verify (c), let \( p \in X \). choose \( n \geq |g(p)| + 1 \), and note that \( \{ x \in X : |g(x) - g(p)| < 1 \} \) is a neighborhood of \( p \) in \( X \) which misses \( L_{-n}(f) \cup L^n(f) \).

Observe next that, to verify (a), it suffices to show that \( f \) has an extension \( f_p \in C(S \cup \{ p \}) \) for every \( p \in X \). (For then \( g : X \to \mathbb{R} \) can be defined by \( g = f \) on \( S \) and \( g(p) = f_p(p) \) for \( p \in X - S \); and since \( S \) is dense in \( X \), \( g \) is continuous [6, 6H].) For the remainder of the proof, we may therefore assume that \( X = S \cup \{ p \} \), with \( p \notin S \).

Assume (b) and (c), and let \( A = \{ s \in \mathbb{R} : p \in \text{cl} L^s(f) \} \), \( B = \{ r \in \mathbb{R} : p \in \text{cl} L_r(f) \} \). Since \( X = \text{cl} S \), (c) implies that there is an \( n \) such that \( p \in \text{cl} L_n(f) \cap \text{cl} L^{-n}(f) \). Hence both \( A \) and \( B \) are nonempty. Moreover, by (b), we have \( s \leq r \) for every \( s \in A \) and every \( r \in B \). Let \( s^* = \sup A \), \( r^* = \inf B \), and note that \( s^*_\leq r^* \). If \( s^* < r^* \), there is \( t \in \mathbb{R} \) with \( s^* < t < r^* \); but then \( p \notin \text{cl}(L_{t}(f) \cup L^t(f)) = \text{cl} S \), a contradiction. Thus \( s^* = r^* \). Define \( g : X \to \mathbb{R} \) by \( g = f \) on \( S \) and \( g(p) = s^* = r^* \). We verify that \( g \) is continuous at each point of \( X \):
Let \( x \in X, \epsilon > 0 \), and \( V = (g(x) - \epsilon, g(x) + \epsilon) \).

Case 1. \( x = p \). Let \( U = X - \text{cl}(L_{g(p)}(f) \cup L_{g(x)}^*(f)) \). Since \( g(p) - \epsilon < s \leq r < g(p) + \epsilon \) for some \( s \in A \) and \( r \in B \), it follows from (b) that \( p \in U \), and clearly \( g(U) \subseteq V \).

Case 2. \( x \in S \). There is an open neighborhood \( W \) of \( x \) in \( S \) with \( f(W) \subseteq (g(x) - (\epsilon/3), g(x) + (\epsilon/3)) \). Write \( W = S \cap G \), with \( G \) open in \( X \). If \( p \notin G \), then \( g(G) = f(W) \subseteq V \), so we may assume that \( p \in G \). Now \( W \subseteq L_{g(x)}^*(f) \), so we have \( p \in \text{cl} G = \text{cl} W \subseteq \text{cl} L_{g(x)}^*(f) \). If \( g(p) < g(x) - (2\epsilon/3) \), there is \( r \in B \) with \( r < g(x) - (2\epsilon/3) \). But then \( p \in \text{cl} L_{g(x)}(f) \subseteq \text{cl} L_{g(x)}^*(f) \), which is contrary to (b). Thus \( g(p) > g(x) - (2\epsilon/3) \), and, similarly, \( g(p) \leq g(x) + (2\epsilon/3) \). We conclude that \( g(G) \subseteq V \), and hence \( g \) is a continuous extension of \( f \).

To complete the proof, note that if \( f \in C^*(S) \) and if \( n > |f| \), then \( L_n(f) \cup L_n^*(f) = \emptyset \), so (c) holds automatically.

**Proof of Theorem 1.** \( A(1) \Rightarrow A(2) \). Assume that \( f = g|S \), with \( g : X \to Y \) continuous. If \( F_1 \) and \( F_2 \) are disjoint closed subsets of \( Y \), then
\[
\text{cl}_X f^{-1}(F_1) \cap \text{cl}_X f^{-1}(F_2) \subseteq g^{-1}(F_1) \cap g^{-1}(F_2) = \emptyset.
\]

Similarly, \( B(1) \Rightarrow B(2) \).

\( A(2) \Rightarrow A(1) \) (resp. \( B(2) \Rightarrow B(1) \)). We may assume that the compact (resp. realcompact) space \( Y = \prod_{a \in I} Y_a \), where \( Y_a = [0,1] \) (resp. \( Y_a = \mathbb{R} \)) for each \( a \in I \) (see [6, 11.12]). Let \( f_a = \left( \text{pr}_a \right)|Y \circ f \), where \( \text{pr}_a \) is the projection of the product \( Y \) of index \( a \). It suffices to show that each \( f_a \) satisfies (b) (resp. (b) and (c)) of Theorem 2. (For then \( f_a \) has a continuous extension \( g_a : X \to Y_a \), the diagonal map \( g = \Delta_{a \in I} g_a : X \to Y \) is continuous, \( g = f \) on \( S \), and \( g(X) = g(\text{cl} S) \subseteq \text{cl} g(S) \subseteq Y \); cf. [4, Lemma 1].) For each \( a \in \mathbb{R} \), let \( Z_a = Y \cap \text{pr}_a^{-1}((\infty, a]) \), \( Z^a = Y \cap \text{pr}_a^{-1}([a, +\infty)) \). Note that \( Z_a \) and \( Z^a \) are zero-sets in \( Y \) and that \( L_{Z_a}(f_a) = f^{-1}(Z_a) \), \( L^a(f_a) = f^{-1}(Z^a) \). It follows from (the zero-set formulation of) either \( A(2) \) or \( B(2) \) that if \( a < b \), then \( L_{Z_a}(f_a) \) and \( L^a(f_a) \) have disjoint closures in \( X \); hence (b) holds in either case. Moreover, \( \bigcap_{n=1}^{\infty} (Z_n \cup Z^*) = \emptyset \), so (the zero-set formulation of) \( B(2) \) implies that \( \bigcap_{n=1}^{\infty} \text{cl}_X (L_{Z_n}(f_a) \cup L^a(f_a)) = \emptyset \). Thus (c) holds, and the proof is complete.

We note that, by a similar argument, Theorem C of [2] is also an easy consequence of Theorem 2.

A subset \( S \) of a topological space \( X \) is \( C^*\)-embedded (resp. \( C\)-embedded) in \( X \) in case every \( f \in C^*(S) \) (resp. \( f \in C(S) \)) has a continuous extension over \( X \). The following corollary (formulated and proved in [6, Theorems 6.4 and 8.6] in the context of Tychonoff spaces; cf. [11]) is an immediate consequence of either Theorem 1 or Theorem 2.

**Corollary.** Let \( S \) be a dense subspace of a topological space \( X \).
A. These are equivalent:
   (1) \( S \) is \( C^*\)-embedded in \( X \).
   (2) Any two disjoint zero-sets in \( S \) have disjoint closures in \( X \).
B. These are equivalent:
   (1) \( S \) is \( C\)-embedded in \( X \).
(2) If a countable family of zero-sets in $S$ has empty intersection, then their closures in $X$ have empty intersection.

It is known that (the closed set formulation of) Theorem 1A holds if $Y$ is merely Tychonoff, provided that $X$ is first countable [2, Theorem D]. For the special case in which $Y$ is a closed subset of $\mathbb{R}$, we can apply Theorem 2 to sharpen this result as follows:

**Theorem 3.** Let $S$ be a dense subspace of a topological space $X$, assume each $p \in X - S$ has a countable base of neighborhoods, let $Y$ be a closed subspace of $\mathbb{R}$, and let $f : S \to Y$ be continuous. Then these are equivalent:

1. $f$ extends continuously over $X$.
2. If $F_1$ and $F_2$ are disjoint countable closed subsets of $Y$, then $f^{-1}(F_1)$ and $f^{-1}(F_2)$ have disjoint closures in $X$.

**Proof.** (1) $\Rightarrow$ (2). This follows as in the proof of A(1) $\Rightarrow$ A(2) of Theorem 1.

(2) $\Rightarrow$ (1). It suffices to show that $f$ (regarded as a function from $S$ into $\mathbb{R}$) has a continuous extension $g : X \to \mathbb{R}$. (For then $g(X) = g(\text{cl } S) \subset \text{cl } g(S) \subset Y$.) We verify that $f : S \to \mathbb{R}$ satisfies (b) and (c) of Theorem 2.

Suppose first that (b) fails. Then for some $a < b$ there exists $p \in \text{cl } L_a(f) \cap \text{cl } L_b(f)$. Obviously $p \in X - S$, so $p$ has a countable base of neighborhoods $\{U_n\}_{n=1}^{\infty}$. Choose $c \in \mathbb{R}$ with $a < c < b$. We shall show that there is a countable closed subset $F_1$ of $\mathbb{R}$ with $p \in \text{cl } f^{-1}(F_1 \cap Y)$ and $F_1 \subset (c, +\infty)$.

**Case 1.** $s^* < +\infty$. For each $n > 0$, we have $a \vee (s^* - (1/n)) < s^*$, so there is $s(n) \in \mathbb{R}$ with $p \in \text{cl } L_{s(n)}(f)$ and $a \vee (s^* - (1/n)) < s(n)$. Moreover, $p \not\in \text{cl } L_{s^*+(1/n)}(f)$, so there exists a point $x_n$ with

$$x_n \in U_n \cap (X - \text{cl } L_{s^*+(1/n)}(f)) \cap L_{s(n)}(f).$$

Let $F_1 = \{f(x_n) : n = 1, 2, \ldots\} \cup \{s^*\}$. Since $|f(x_n) - s^*| < 1/n$, we have $f(x_n) \to s^*$, and hence $F_1$ is closed in $\mathbb{R}$. Clearly $p \in \text{cl } f^{-1}(F_1 \cap Y)$ and $F_1 \subset (c, +\infty)$.

**Case 2.** $s^* = +\infty$. Construct a sequence $(x_n)_{n=1}^{\infty}$ as follows: Pick $x_1 \in U_1 \cap L^b(f)$; and if $x_1, \ldots, x_{n-1}$ have already been chosen with $x_i \in U_i \cap L^b(f)$ and $f(x_i) > f(x_{i-1}) \lor i$ ($i = 2, \ldots, n - 1$), choose $s \in \mathbb{R}$ with $p \in \text{cl } L^b(f)$ and $f(x_{n-1}) \lor n < s$, and pick $x_n \in U_n \cap L^b(f)$. Then $(f(x_n))_{n=1}^{\infty}$ is strictly increasing and $f(x_n) \to +\infty$, so $F_1 = \{f(x_n) : n = 1, 2, \ldots\}$ is closed in $\mathbb{R}$. Moreover, $x_n \in U_n \cap L^b(f)$ for all $n$, so we have $p \in \text{cl } f^{-1}(F_1 \cap Y)$ and $F_1 \subset (c, +\infty)$.

Similarly, there is a countable closed subset $F_2$ of $\mathbb{R}$ with

$$p \in \text{cl } f^{-1}(F_2 \cap Y) \quad \text{and} \quad F_2 \subset (-\infty, c).$$

Thus (2) fails.

Suppose next that (c) of Theorem 2 fails. Then there exists $p \in \bigcap_{n=1}^{\infty} \text{cl}_X(L_{n-1}(f) \cup L_n(f))$, and clearly $p \in X - S$. Let $(U_n)_{n=1}^{\infty}$ be a countable base of neighborhoods at $p$ with $U_n \subset U_{n+1}$ for each $n$. Pick

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$x_1 \in U_1 \cap (L_{-1}(f) \cup \overline{L^1(f)})$; and if $x_1, \ldots, x_{n-1}$ have already been chosen with $x_i \in U_i \cap (L_{-i}(f) \cup \overline{L^i(f)})$ and $|f(x_i)| > |f(x_{i-1})|$ \forall (i = 2, \ldots, n - 1), let $m(n)$ be the least integer $\geq |f(x_{n-1})|$ $\forall n$, and pick $x_n \in U_n \cap (L_{-m(n)}(f) \cup \overline{L^{m(n)}(f)})$. We thus construct a sequence $(x_n)_{n=1}^\infty$ with $x_n \in U_n \{ |f(x_n)| \}_{n=1}^\infty$ strictly increasing, and $|f(x_n)| \to \infty$. Let

$$F_1 = \{ r \in \mathbb{R} : |r| = |f(x_n)| \text{ for some } n, n \text{ odd} \},$$

$$F_2 = \{ r \in \mathbb{R} : |r| = |f(x_n)| \text{ for some } n, n \text{ even} \}.$$

Then $F_1$ and $F_2$ are disjoint countable closed subsets of $\mathbb{R}$ with

$$p \in \cl f^{-1}(F_1 \cap Y) \cap \cl f^{-1}(F_2 \cap Y),$$

so (2) fails once again. The proof is therefore complete.

We leave open the question of possible generalizations of Theorem 3 (for Tychonoff spaces $Y$ that are not necessarily closed subspaces of $\mathbb{R}$).

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