GENERALIZED MORSE SEQUENCES
ON n SYMBOLS

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Abstract. A class of bisequences on n symbols is constructed which includes the generalized Morse sequences introduced by Keane. The topological structure and endomorphisms of the resulting minimal symbolic flows are described.

Introduction. We construct a class of bisequences on s symbols (s ≥ 2) which contains the generalized Morse sequences on two symbols described by Keane in [9]. The orbit-closures of these sequences in the shift dynamical system on s symbols are point-distal symbolic flows, and we consider their topological structure. In our main theorems, we describe the maximal equicontinuous factor of such a flow; we prove that the symbolic flow is an isometric extension of an almost automorphic extension of its maximal equicontinuous factor; and we determine all endomorphisms of the flow. These theorems generalize results of Coven, Keane, and the author on substitution minimal sets. For basic definitions, the reader is referred to [3], [5], and [11]. The author would like to thank the referee for a helpful suggestion regarding the proof of Theorem 7.

1. Construction. Let s be an integer greater than 1, and let S = {0, 1, ..., s - 1}. B_k will denote the set of k-blocks over S, X the set of sequences over S (i.e., functions from the nonnegative integers to S), and Y the set of bisequences over S. If A ∈ B_k, C ∈ B_m, AC ∈ B_{k+m} is defined by AC = A(0)A(k - 1)C(0)C(m - 1). For x an element of B_k, X, or Y, x(j,m) will denote the m-block x(j)x(j + 1)...x(j + m - 1). If A ∈ B_k, let L(A) = k.

Now we take G = {σ_0, σ_1, ..., σ_{s-1}} to be any subgroup of the group of permutations of {0, 1, ..., s - 1}, where σ_0 is the identity. Thus σ_i may be considered as a function from B_k, X, or Y to itself. If A ∈ B_j, C ∈ B_k, define

A × C = σ_C(0)Aσ_C(1)A ... σ_C(k-1)A ∈ B_{jk}.

For each j ≥ 0, let m_j ≥ 2, and let b_j be an element of B_{m_j} with b_j(0) = 0. Then we may define an element of X as follows:

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\[ x = \cdots ((b_0 \times b_1) \times b_2) \times \cdots. \]

For \( t \geq 0 \), let \( c_t = (\cdots ((b_0 \times b_1) \times b_2) \times \cdots b_t) \); let \( n_t = L(c_t) = m_0 \cdots m_t \).

We observe that for each \( t \),

\[ x = c_t \sigma_i(c_t) \sigma_j(c_t) \cdots \]

for some sequence \( i_1, i_2, \ldots \) satisfying \( 0 i_1 i_2 \cdots i_{m_t+1} = b_{t+1} \).

We assume that both \( b_0 \) and, for each \( t \), the associated sequence \( i_1 i_2 \cdots \) in (1) contain every symbol in \( S \).

**Lemma 1.** \( x \) is a periodic sequence if and only if, for some \( t \), the sequence \( i_1 i_2 \cdots \) in (1) is periodic of period \( s \).

For \( s = 2 \), this is Lemma 1 of [9]. The proof in the general case is a straightforward, if somewhat tedious, adaptation of that in [9].

We let \( T: \Omega \to \Omega \) denote the shift transformation: \( (T \omega)(n) = \omega(n + 1) \) \((n \in \mathbb{Z})\).

**Proposition 2.** There is an almost periodic point \( \omega \) in the shift dynamical system \((\Omega, T)\) with \( \omega(n) = x(n) \) \((n \geq 0)\).

**Proof.** Let \( k \geq 0 \), and choose \( t \) so that \( n_t \geq k \); by (2) we may find \( u \) so that each \( n_t \)-block \( \sigma_j(c_t) \) appears in \( c_u \). Now by (1), \( x = c_u \sigma_i(c_u) \cdots \). Thus any \( 2n_u \)-block of \( x \) contains \( \sigma_j(c_u) \) for some \( j \); but \( c_u \) contains the \( n_t \)-block \( \sigma_i^{-1}(c_t) \). Thus every \( 2n_u \)-block of \( x \) contains \( x(0,k) \). It is now an easy matter to extend \( x \) to an almost periodic sequence \( \omega \). Q.E.D.

We assume from now on that \( \omega \) is a fixed, nonperiodic, almost periodic sequence which extends \( x \). We call \( \omega \) a generalized Morse sequence (though in [9] the term is reserved for sequences of this type which are strictly transitive). We denote by \( X_\omega \) the orbit-closure of \( \omega \) under \( T \).

2. A basic lemma on the block structure of \( x \). For \( t \geq 0 \), and \( A \) a \( k \)-block of \( x \), \( A \) is said to be determined to order \( t \) if, whenever \( A = x(n,k) = x(m,k) \), then \( m = n \) \((\text{mod } n_t)\).

**Lemma 3.** For any \( t \), there is a \( k \) so that every \( k \)-block of \( x \) is determined to order \( t \).

**Proof.** It is sufficient to find some \( t \) for which the statement holds. Choose \( t \) large enough so that \( c_t = ABC \), where \( A \) and \( C \) both contain every symbol in \( S \), and \( \max(L(A), L(C)) \leq n_t/s \). Now, by redefining \( m_0 \), we may as well assume \( t = 0 \).

We show that some \( c_u \) is determined to order 0. If not, then for each \( u \), \( c_u = x(a_u, m_u) \) for some \( a_u \neq 0 \) \((\text{mod } m_0)\). By considering a subsequence, we may assume that for some \( a \) \((0 < a < n_0)\), \( a_u = n_0 - a \) \((\text{mod } n_0)\) for each \( u \). This implies that \( x = D \sigma_i(a_0) \sigma_j(a_0) \cdots \), where \( D \in B_{a_t} \) and any initial portion of \( \sigma_{j_1}(a_0) \sigma_{j_2}(a_0) \cdots \) appears in \( x \) beginning at some position equal to 0 \((\text{mod } n_0)\). We consider two cases.

(i) \( m_0/s < a < (1 - 1/s)n_0 \). We have \( x = c_0 \sigma(a_0) \sigma_i(a_0) \cdots \). Consider the sequence \( j_1 i_1 j_2 i_2 \cdots \), and take any \( j_k \). The last \( a \)-block of \( \sigma_{j_k}(a_0) \) is the first \( a-
block of $\sigma_k(c_0)$, and it contains $\sigma_j(C)$ since $L(C) \leq n_0/s$. Thus it contains every symbol in $S$. This implies that $i_k$ is determined uniquely by $j_k$. Similarly, given $i_k, j_k$ is determined. Thus the sequence $j_1 i_1 \cdots$ is periodic, since every symbol has a unique successor; but this contradicts the fact that $x$ is not periodic.

(ii) $a < n_0/s$ or $a > (1 - 1/s)n_0$. Again $c_0 \sigma_{i_1}(c_0) \cdots = D \sigma_{j_1}(c_0) \cdots$, where $D \in B_n$. This implies that for $k, m \geq 0$, $x(kn_0 + ma, n_0)$ is a block of the form $\sigma(c_0)$. It follows that if $d$ is the greatest common divisor of $n_0$ and $a$, then for $k \geq 0, x(kd, n_0)$ is of the form $\sigma(c_0)$. Now let the distinct $d$-blocks of the form $x(kd, d)$ be denoted $A_1, A_2, \ldots, A_r$. Then $x = A_1 A_2 \cdots$, and for each $k$ the nonperiodic sequence $i_1 i_2 \cdots$ contains at least $k + 1$ distinct $k$-blocks. Letting $k = n_0/d$, we obtain $k > s$, using our assumption on $a$. We have shown that there are more than $s$ distinct $n_0$-blocks of the form $x(kd, n_0)$; but every such block is $\sigma(c_0)$ for some $i$. This contradiction completes the proof. Q.E.D.

For $s = 2$, the above lemma is approximately Lemma 5 of [9].

We list some simple consequences of Lemma 3.

**Lemma 4.** (a) For each $t$, $\omega(-n_t, n_t) = \sigma(c_i)$ for some $i$.

(b) If $y \in X$, and $T^{j_k y}$ converges, then for each $t, j_k - j_m = 0 \pmod{n_t}$ for all sufficiently large $k$ and $m$.

3. **Equicontinuous factors of** $(X^\omega, T)$. We denote by $(Z(k), 1)$ the minimal rotation $z \to z + 1$ on the cyclic group $Z(k)$ of order $k$. If $a = (a_0, a_1, \ldots)$, where $a_i \geq 2$, and $d_i = a_0 a_1 \cdots a_i$, we let $(\Delta(a), 1)$ be the minimal equicontinuous flow $z \to z + 1$ on the group $\Delta(a)$ of $a$-adic integers—that is the inverse limit of the groups $Z(d_i)$. (Here "1" means the element $(1, 1, \ldots) \in \Delta(a)).$

**Proposition 5.** There is a flow homomorphism $f$ from $(X^\omega, T)$ to $(\Delta(m), 1)$, where $m = (m_0, m_1, \ldots)$, such that if $z = (z_0, z_1, \ldots) \in \Delta(m)$, $f(y) = z$ if and only if $y(-z_t, n_t)$ is of the form $\sigma(c_i)$ for every $t$. For $i \in S$, $y \in X$, $f(a_i y) = f(y)$. If $z$ is not in the orbit of $0$ in $\Delta(m)$ and $y \in f^{-1}(z)$, $f^{-1}(z) = \{\sigma_i y : i \in S\}$. Q.E.D.

**Corollary 6.** If $z \in \Delta(m)$ is not an integer, any point in $f^{-1}(z)$ is a distal point. Hence $(X^\omega, T)$ is point-distal.
Theorem 7. \((X_\omega^*, T)\) is isomorphic to \((\Delta(m'), 1)\), for some \(m' = (m'_0, m_1, m_2, \ldots)\), where \(m'_0 = m_0 r\) for some divisor \(r\) of \(s\). \((X_\omega, T)\) is a proper AI extension of \((X_\omega^*, T)\).

Proof. We first show that \((X_\omega, T)\) is an AI extension of \((\Delta(m), 1)\). Let \((Y, T) = (X_\omega, T)/G\) (using Proposition 5). We then have \((X_\omega, T) \xrightarrow{\delta} (Y, T) \xrightarrow{\pi} (\Delta(m), 1)\), where \(hg = f\), and \(h^{-1}(z)\) is a single point if \(z\) is not in the orbit of 0. If we define \(R = \{(x_1, x_2); g(x_1) = g(x_2)\}\) by \(R(x, y) = 0\) if \(x = y\), \(R(x, y) = 1\) if \(x \neq y\), then \(R\) is continuous. Thus \((X_\omega, T)\) is an isometric extension of \((Y, T)\).

Now from [10, Theorem 8.11], \((X_\omega, T)\) is an AI extension of \((X_\omega^*, T)\). We obtain the diagram

\[
(X_\omega^*, T) \xrightarrow{\rho} (W, T) \xrightarrow{\pi} (X_\omega^*, T) \xrightarrow{\delta} (\Delta(m), 1),
\]

where \(\pi \circ \rho = f\), \(\pi\) is \(r\)-to-one for some divisor \(r\) of \(s\), and \(\rho\) is \(s/r\)-to-one. Thus, using the criterion for two groups \(\Delta(m)\) and \(\Delta(m')\) to be isomorphic, we see that \(X_\omega^*\) is isomorphic to \(\Delta(m')\), where \(m'\) is of the desired form. Finally, it can be seen that \(r \neq s\), so that \(\rho\) is not 1-1. Q.E.D.

Corollary 8. If either (a) \(s\) is prime; or (b) every prime factor of \(s\) appears in infinitely many \(m_i\)'s, then \((X_\omega^*, T)\) is isomorphic to \((\Delta(m), 1)\).

It is possible, however, to construct for each nonprime \(s\) examples where \((X_\omega^*, T)\) is not isomorphic to \((\Delta(m), 1)\).

Corollary 9. If \(Z\) is any infinite, compact, zero-dimensional, monothetic group, 1 is a generator for \(Z\), and \(s \geq 2\), there is a generalized Morse sequence \(\omega\) on \(s\) symbols with \((X_\omega^*, T) \cong (Z, 1)\).

Proof. This is a simple consequence of the fact that any such \(Z\) is isomorphic to \(\Delta(m)\) for some \(m [7]\). Q.E.D.

4. Endomorphisms of \((X_\omega, T)\).

Theorem 10. If \(\psi\) is an endomorphism of \((X_\omega, T)\), then \(\psi = T^k \sigma_m\) for some \(k \in \mathbb{Z}, m \in S\).

Proof. We let \(B_j(\omega)\) denote the set of \(k\)-blocks of \(\omega\). It is well known that for some integer \(p\) and some \(g : B_j(\omega) \to S\), the map \(\phi = T^p \psi\) is the block map \(g_\infty\) (see [6]). For \(n \geq 1\), let \(g_n : B_{j+n-1}(\omega) \to B_n(\omega)\) be the function induced by \(g\), and choose \(n \geq 2n_0\) with \(j + n - 1 = n\). Now for some unique \(r (0 \leq r < n)\), each block \(\phi(\omega)(r + in_1, n_1)\) is of the form \(\sigma_{j_1}(c_i)\). We assume \(r \geq 2n_0\); the other case is proved similarly. Then for each \(i\), \(\phi(\omega)(r + (i - 1)n_1, n_1)\) is determined by \(\omega(in_1, n_1)\). Let \(\phi_i = T^{-n}\phi_i\). Then given either of the blocks \(\phi_i(\omega)(in_1, n_1)\) and \(\omega(in_1, n_1)\), the other is determined. Then for \(u \geq t\), \(\phi(\omega)(in_1, n_1)\cdot (in_1, n_1)\) is of the form \(\sigma_k(c_u)\). (Otherwise, for some \(a\) with \(0 < a < n_u\) and \(a = 0 (mod n_1)\), we have \(\phi_i(\omega)(a + in_1, n_1) = \sigma_k(c_u)\) for each \(i\), from which it follows that if \(\omega(in_1, n_1) = \sigma_j(n_1)\), the sequence \(j_0k_0j_1k_1 \cdots\) is periodic.) Hence, for some \(m, \phi(\omega)(0, n_u) = \sigma_m(0, n_u)\) for each \(u\), and thus \(\phi = \sigma_m\). Q.E.D.

5. Remarks. We comment briefly on the almost automorphic flows \((Y, T)\) obtained in Theorem 7, in the special case when \(G\) is the group generated by
the cyclic permutation $\sigma_i : j \rightarrow j + 1 \pmod{s}$. $(Y, T)$ is isomorphic to a symbolic flow, and the map $g$ may be defined by $(gx)(n) = x(n) + x(n + 1) + \cdots + x(n + s - 1) \pmod{s}$. This is similar to the construction discussed in [2] and [5]. $(Y, T)$ is always a strictly ergodic flow, and thus using the results of [9], we obtain examples of a two-to-one group extension of a strictly ergodic flow which is not strictly ergodic. It is also possible to show that the only endomorphisms of $(Y, T)$ are powers of the shift.

Certain special cases of this construction have been discussed extensively. By taking $b_0 = b_1 = \cdots$, and $G$ the cyclic group above, we obtain a substitution minimal set. If $s = 2$, every continuous substitution minimal set can be obtained this way, and hence our main theorems generalize results of Coven and Keane in [1] and [2]. In another special case ($s = 4$), it is possible to obtain the strictly transitive sequence constructed by Kakutani in [8].

REFERENCES


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