THE FUNDAMENTAL THEOREM OF ALGEBRA
ON RATIONAL $H$-SPACES

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Abstract. A form of the Fundamental Theorem of Algebra is proven for
$\Gamma$-$H$ structures.

0. Introduction and summary of results. It is a curiosity that while the
Fundamental Theorem of Algebra is algebraic in content and statement, its
proof is topological. We show here that the structure of a rational $H$-space, or
$\Gamma$ structure is sufficient to prove a form of the theorem.

In § 1 we construct a rational $H$-space structure on the Stiefel manifold $V_{3,2}$
which we believe has not been explicitly exhibited before. In § 2 we prove that
any homotopy associative rational $H$-space is actually an $H$-space.

1. Rational $H$-spaces or $\Gamma$ structures. A rational $H$-space is a triple $(X,m,e)$
consisting of a connected space $X$, a product map $m : X \times X \to X$ which
preserves the basepoint $e$, and which satisfies the following condition. Let
$q_i : X \to X \times X$ be inclusion into the $i$th factor, $i = 1, 2$. Then $m_l = mq_l$ and
$m_r = mq_r$ must induce automorphisms of $H^*(X; \mathbb{Q})$.

If $m_l \simeq m_r \simeq \text{id}$, then $e$ is a homotopy identity and $(X,m,e)$ is actually an
$H$-space.

For notational simplicity we will follow Hopf [4] and refer to rational $H$-
spaces as $\Gamma$ structures.

Examples. (1) (Hopf [4]). For an odd dimensional sphere define $m : S^n \times S^n
\to S^n$ by $m(p,q) = q$ reflected through the orthogonal complement of $p$. Then
$m_l^*(z) = -z$ and $m_r^*(z) = 2z$ where $z$ generates $H^*(S^n)$.

(2) Consider the Stiefel manifold $V_{n,2}$ of 2 frames in $\mathbb{R}^n$ for odd $n$. $V_{n,2}$
can be fibered as an $n - 2$ sphere bundle over $S^{n-1}$ [5]. Since $\Pi_{2n-3}(S^{n-1})$ has a
cyclic infinite subgroup [3, p. 74], and $\Pi_k(S^{n-2})$ is finite whenever $k \neq n - 2
[7, p. 515]$, it is clear from the long exact sequence of the fibration that
$\Pi_{2n-3}(V_{n,2})$ has a cyclic infinite subgroup, with generator, say $g$.

Let $\mu$ be a generator of $H^{2n-3}(S^{2n-3}, \mathbb{Z})$. Then $g_*(\mu) = h([g])$ where
$h : \Pi_{2n-3}(V_{n,2}) \to H_{2n-3}(V_{n,2}; \mathbb{Z})$ is the Hurewicz homomorphism. Since
$H^*(V_{n,2}; \mathbb{Q}) = \Lambda(z_{2n-3}, 1)$, an exterior algebra on odd dimensional generators
[6], we can apply a result of Arkowitz and Curjel [1] to conclude that
$g_*(\mu) \neq 0$.

The cell structure of $V_{n,2}$ is known to be $S^{n-2} \cup e^{n-1} \cup e^{2n-3}$. Let $\rho : V_{n,2}$

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\[ S^{2n-3} \text{ be the collapsing map. Then the composition} \]
\[ V_{n,2} \times V_{n,2} \xrightarrow{\partial \times \rho} S^{2n-3} \times S^{2n-3} \xrightarrow{m} S^{2n-3} \xrightarrow{g} V_{n,2} \]
\[ \text{is a } \Gamma \text{ structure on } V_{n,2} \text{ where } m \text{ is Hopf's } \Gamma \text{ structure.} \]

It is well known that if a CW complex \( X \) admits a \( \Gamma \) structure, then \( H^*(X, \mathbb{Q}) = \Lambda_{r \in \mathbb{N}}(z_r) \otimes P_{\mu \in \mathbb{M}}(x_\mu) \) where \( \Lambda(z_r) \) is an exterior algebra on odd dimensional generators and \( P(x_\mu) \) is a polynomial algebra on even dimensional generators. Furthermore, for \( \gamma \in H^*(X, \mathbb{Q}) \),
\[ m^*(\gamma) = m_l^*(\gamma) \otimes 1 + \sum y' \otimes y'' + 1 \otimes m_r^*(\gamma) \]
with \( \deg y' > 0 \).

2. Words on \( \Gamma \) structures. The product \( m \) induces a binary operation, called convolution, on the set of maps from \( X \) to itself. \( f \circ g = m \circ f(x) \circ \Delta \) where \( \Delta(x) = (x, x) \), the usual diagonal. A word on \( X \) is the convolution of a finite number of identity and constant maps. The identity map on \( X \) will be denoted by \( 1 \), the constant map at \( p \) by \( \omega_p \) or simply \( \omega \). Other identity maps will be denoted by \( id \).

Theorem 1.1. If \((X, m, e)\) is a homotopy associative \( \Gamma \) structure, then \( X \) admits an \( H \) structure.

Proof. It suffices to show that \( m_r^* = m_l^* = id \). Then, by Whitehead's theorem, \( m_r \) and \( m_l \) are homotopy equivalences and \( m \circ (m_l^{-1} \times m_r^{-1}) \) is an \( H \) structure.

If \( m \) is homotopy associative, then, in particular, \((\omega \cdot \omega)^* \cdot 1 = \omega^* \cdot (\omega \cdot 1)^* \).

Direct computation shows that the left-hand side is \( m_r^* \) and the right side is \( m_r^* \circ m_l^* \). A similar computation holds for \( m_l^* \).

We adopt the technical convention that all words will be spelled correctly. That is, convolution is always from the left, i.e.,
\[ g_1 \cdot g_2 \cdot \ldots \cdot g_r = g_1 \cdot (g_2 \cdot (\ldots (g_{r-1} \cdot g_r))) \cdot \ldots \]
and no two consecutive maps are constants. (This is always possible since \( \omega_p \cdot \omega_q = \omega_{p(q)} \)).

Let \( \{y_j\} \) denote the canonical basis for \( H^*(X; \mathbb{Q}) \) as a vector space over \( \mathbb{Q} \) corresponding to \( \{z_p\}_{p \in \mathbb{N}} \) and \( \{x_\mu\}_{\mu \in \mathbb{M}} \). A typical element is of the form \( y_j z_{p_1} z_{p_2} \ldots z_{p_r} x_{\mu_1} x_{\mu_2} \ldots x_{\mu_t} \). For each basis element \( y_j \) we define an integer
\[ \lambda(y_j) = \max(\text{degree } z_{p_1}, \text{degree } x_{\mu_i}, i = 1, \ldots, k; j = 1, \ldots, r). \]

Let \( G(X) \) be the vector space over \( \mathbb{Q} \) spanned by \( \{z_p\}_{p \in \mathbb{N}} \cup \{x_\mu\}_{\mu \in \mathbb{M}} \). Define \( \alpha(z_p) \) and \( \beta(z_p) \) to be the summand of \( m_l^* (z_p) \) and \( m_r^* (z_p) \) which lives in \( G(X) \), i.e., if \( m_r^* (z_p) = \sum a_i z_{p_i} + \sum a_j y_j \), \( a_i, a_j \in \mathbb{Q} \) and \( \lambda(y_j) < \text{degree } z_{p_r} \), then \( \alpha(z_p) = \sum a_i z_{p_i} \).

Associated to every word \( f \) on \( X \) we define an automorphism of \( G(X) \) called the exponent of \( f \), \( \epsilon(f) \).

\[ \epsilon(f) = \begin{cases} id & \text{if } f = 1, \\ 0 & \text{if } f = \omega, \\ \alpha \circ \epsilon(g) \circ \beta & \text{if } f = 1 \cdot g, \\ \epsilon(g) \circ \beta & \text{if } f = \omega \cdot g. \end{cases} \]
If \((X,m,e)\) is an \(H\)-space, then \(\alpha = \beta = \text{id}\) and \(\epsilon(f)\) coincides with the definition of exponent given by R. F. Brown [2].

**Lemma 2.1.** If \(f : X \to X\) is a word, then \(f^*(z) = \epsilon(f)z + \sum a_j y_j\) where \(\lambda(y_j) < \deg z\).

**Proof.** The result is clear if \(f = \iota = \omega\). We proceed by induction on the number of maps convoluted to form \(f\). If \(f = \iota \cdot g\), then \(f^*(z) = \alpha(z) + \sum a_j y_j + g^*(\beta(z))\), \(\beta(z) = \sum b_i z_i\), \(b_i \in \mathbb{Q}\), \(\deg z_i = \deg z\). Hence

\[
g^*(\beta(z)) = \sum b_i g^*(z_i) = \sum b_i (\epsilon(g)z_i) + \sum a_j y_j
\]

with \(\lambda(y_j) < \deg z_i\). The case \(f = \omega \cdot g\) is a similar computation. □

3. \(\Gamma-H\) structures. Following [2] we define \(C^0(X)\), the open cone on \(X\), as \(X \times [0, \infty)\), \(X \times \{0\}\) collapsed to a point. Square brackets will be used to denote equivalence classes and \([x, 0]\) will be written as 0. Any \(\Gamma\) structure \((X,m,e)\) can be extended to a \(\Gamma\) structure \((C^0(X),c(m),[e, 1])\) by defining \(c(m)([x, t], [y, s]) = [m(x, y), ts]\).

A \(\Gamma-H\) structure \((X,m,e,\eta)\) is a \(\Gamma\) structure \((X,m,e)\) together with an \(H\) structure on \(C^0(X)\) for which 0 is the basepoint, i.e., \(\eta : C^0(X) \times C^0(X) \to C^0(X)\) and \(\eta([x, t], 0) = \eta(0, [x, t]) = [x, t]\). We will denote the binary operation \(\eta\) induces on words of the \(\Gamma\) structure \((C^0(X),c(m),[e, 1])\) by \(\varphi\). A polynomial on \(C^0(X)\) is a map from \(C^0(X)\) to itself of the form \(g_1 \varphi g_2 \varphi \cdots \varphi g_k\), where \(g_i\) is a word on \((C^0(X),c(m),[e, 1])\). \([x, t]\) is a root of a polynomial if \(f([x, t]) = 0\).

**Example 3.1.** Let \(X = S^1\) considered as complex numbers of unit norm, \(m\) is standard complex multiplication, \(\eta\) is usual addition. \(\omega_\alpha \cdot \iota^n \varphi \omega_{\alpha-1} \cdot \iota^{n-1}\), \(\varphi \cdots \varphi \omega_\alpha\) is the familiar polynomial \(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0\).

Given a polynomial \(f = g_1 \varphi g_2 \varphi \cdots \varphi g_k\), define a polynomial

\[
\psi(s, g_i) \varphi \cdots \varphi \psi(s, g_{i-1}) \varphi g_i \varphi \psi(s, g_{i+1}) \varphi \cdots \varphi \psi(s, g_k),
\]

i.e. apply \(\psi(s, \ )\) to each word except \(g_i\) where

\[
\psi(s, g)([x, t]) = \theta(s, g[x, t]) \quad \text{and} \quad \theta(s, [x, t]) = [x, st].
\]

The polynomial \(f\) is admissible if \(f_{i,s}\) is proper for some \(i\) uniformly in \(s\), i.e. \(f_{i,s}\) extends to the suspension \(SX\) viewed as the one point compactification of \(C^0(X)\). Not all nonconstant words are admissible.

**Example 3.2.** Consider the quaternions as \(C^0(S^3)\). Then the polynomial \(\omega_{[1]} \varphi \iota \cdot \omega_{[1]} \varphi \omega_{[1]} \cdot \iota\) is not admissible and does not have a root.

**Lemma 3.3.** If \(f = g_1 \varphi \cdots \varphi g_k\) is an admissible polynomial with \(f_{i,s}\) proper, then \(\tilde{f} \simeq \tilde{g}_i\), where \(\tilde{f}\) denotes the extension to \(SX\).

**Proof.** Define \(H : SX \times I \to SX\) by \(H([x, t], s) = \tilde{f}_{i,s}[x, t]\).

A word \(g\) on \((X,m,e)\) is compatible with \(m\) if \(\epsilon(g) \neq 0\). A polynomial \(f = g_1 \varphi \cdots \varphi g_k\) is compatible if \(f\) is admissible and \(g_i\) is compatible with \(c(m)\).

Define a word \(\tilde{g}_i\) on \((X,m,e)\) by \(\tilde{g}_i(x) = g_i[x, 1]\). Then by 3.3, \(\tilde{f} \simeq \tilde{g}_i \simeq \sum \tilde{g}_i\).

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Example 3.4. Let $m : S^1 \times S^1 \to S^1$ be given by $m(e^{i\theta}, e^{i\varphi}) = e^{i(\theta - \varphi)}$. The polynomial $i^2 \varphi \omega_{[e^{i\varphi}/2, 1]}$ is admissible but not compatible. In this case $g_i = i^2$ and $\epsilon(g) = 1 - 1^i = 0$. It is not hard to show that $f$ is rootless.


Theorem. $(X, m, e, n)$ is a $\Gamma$-H structure with $H^*(X; \mathbb{Q}) = \Lambda\{z_r\} \otimes P\{x_\mu\}$. Every polynomial on $C^0(X)$ which is compatible with $m$ has a root.

Proof. Suppose $f = g_1 \varphi g_2 \varphi \cdots \varphi g_k$. By compatibility $\bar{f} \approx \bar{g}_i$ and $\epsilon(\bar{g}_i) \neq 0$. $\bar{g}_i^*(z_r) = \epsilon(\bar{g}_i)z_r + \sum a_jy_j \neq 0$ by Lemma 2.1. Then $\bar{f}^*o(z_r) = S\bar{g}_i^*(\sigma z_r) = \sigma g_i^*(z_r) \neq 0$ where $\sigma : H^*(X) \to H^*(SX)$ is the suspension homomorphism.

On the other hand if $\bar{f}$ is not onto then its image will be contained in a contractible space and $\bar{f}^* = 0$.

Note this generalizes Theorem 2 of [2].

References


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