A CHARACTERIZATION OF $B$-SLOWLY VARYING FUNCTIONS

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Abstract. A measurable function $\varphi > 0$ that satisfies the limit condition
\[ \lim_{x \to \infty} \frac{\varphi(x + t\varphi(x))}{\varphi(x)} = 1 \]
for all $t$ is said to be $B$-slowly varying. If $\varphi$ is continuous, this limit is shown to hold uniformly for $t$ in compact sets, and an integral representation is derived.


Theorem 1. Let $\varphi$ be a positive function satisfying
\[ \lim_{x \to \infty} \frac{\varphi(x + t\varphi(x))}{\varphi(x)} = 1 \quad \text{for each fixed } t, \]
and
\[ \lim_{x \to \infty} \frac{\varphi(x)}{x} = 0. \]

Let $K \in L^1$ satisfy $\int_{-\infty}^{\infty} K(t)e^{-i\lambda t} dt \neq 0$ for each real $\lambda$, and let $f \in L^\infty$. If there exists a constant $A$ such that
\[ \lim_{x \to \infty} \int_{-\infty}^{\infty} f(t)K\left[ \frac{x - t}{\varphi(x)} \right] \frac{dt}{\varphi(x)} = A \int_{-\infty}^{\infty} \varphi(x) \varphi(x) \]
then
\[ \lim_{x \to \infty} \int_{-\infty}^{\infty} f(t)H\left[ \frac{x - t}{\varphi(x)} \right] \frac{dt}{\varphi(x)} = A \int_{-\infty}^{\infty} \varphi(x) \varphi(x) \]
for all $H \in L^1$.

Wiener's Theorem is the special case, $\varphi \equiv 1$, of Theorem 1.

Definition. A positive, measurable function $\varphi$ that satisfies (1) is $B$-slowly varying, and $B$ denotes the class of all such functions. If $\varphi$ satisfies (1) uniformly for $t$ in every bounded interval $(a,b)$, then $\varphi$ is uniformly $B$-slowly varying, denoted $\varphi \in B_u$.

Theorem 2. If $\varphi \in B$ is continuous, then $\varphi \in B_u$.

Proof. We prove this for $t$ between zero and one. The argument for an arbitrary interval $(a,b)$ follows similarly.

Suppose that $\varphi$ is not uniformly $B$-slowly varying. Then there is an $\epsilon \in (0,1)$ and sequences $\{t_n\} \subset (0,1)$ and $\{x_n\}$ tending to infinity, such that...
\[(5) \quad \left| \frac{\varphi(x_n + t_n \varphi(x_n))}{\varphi(x_n)} - 1 \right| \geq \epsilon \quad (n = 1, 2, \ldots).\]

The function \(f_n(t) = \frac{\varphi(x_n + t \varphi(x_n))}{\varphi(x_n)} - 1\) is continuous and \(\lim_{n \to \infty} |f_n(t)| = 0\) for fixed \(t\). So there is an integer \(N\) and a sequence \(\{\lambda_n\} \subseteq (0, 1)\) such that

\[(6) \quad \left| \frac{\varphi(y_n)}{\varphi(x_n)} - 1 \right| = \epsilon \quad (n \geq N),\]

where \(y_n = x_n + \lambda_n \varphi(x_n)\). Set

\[
V_n = \{\lambda \in (0, 2 + \epsilon) : |f_n(\lambda)| < \epsilon/2\},
\]

\[
W_n = \{\mu \in (0, 1) : \left| \frac{\varphi(y_n + \mu \varphi(y_n))}{\varphi(y_n)} - 1 \right| < \epsilon/2(1 + \epsilon)\},
\]

\[
W'_n = \{\lambda = \lambda_n + \mu \varphi(y_n)/\varphi(x_n) : \mu \in W_n\}.
\]

These sets are Lebesgue measurable, with

\[(7) \quad \lim_{n \to \infty} \mathfrak{M}(V_n) = 2 + \epsilon, \quad \lim_{n \to \infty} \mathfrak{M}(W_n) = 1.\]

(Korevaar, Van Aardenne-Ehrenfest, DeBruijn [4] cite De Le Vallee Poussin [7] for this. One may also apply Egoroff's Theorem to \(f_n(t)\).) \(W'_n \subseteq (0, 2 + \epsilon)\), and \(\mathfrak{M}(W'_n) \geq (1 - \epsilon)\mathfrak{M}(W_n)\) so that

\[(8) \quad \lim_{n \to \infty} \inf \mathfrak{M}(W'_n) \geq 1 - \epsilon.\]

For \(\lambda \in W'_n\),

\[(9) \quad \left| \frac{\varphi(x_n + \lambda \varphi(x_n))}{\varphi(x_n)} - \frac{\varphi(y_n)}{\varphi(x_n)} \right| = \left| \frac{\varphi(y_n)}{\varphi(x_n)} \right| \left| \frac{\varphi(y_n + \mu \varphi(y_n))}{\varphi(y_n)} - 1 \right| < \frac{\epsilon}{2}\]

so that

\[(10) \quad \left| \frac{\varphi(x_n + \lambda \varphi(x_n))}{\varphi(x_n)} - 1 \right| > \frac{\epsilon}{2}\]

and, in particular, \(\lambda \notin V_n\). Thus \(V_n \cap W'_n = \emptyset\), \(V_n, W'_n \subseteq (0, 2 + \epsilon)\), so that

\[(11) \quad 2 + \epsilon \geq \lim_{n \to \infty} \mathfrak{M}(V_n \cup W'_n) \geq \lim_{n \to \infty} (\mathfrak{M}(V_n) + \mathfrak{M}(W'_n)) \geq 3,
\]

or \(\epsilon \geq 1\), which is impossible.

**Slowly varying functions.**

**Definition.** A positive, measurable function \(g\) is slowly varying if it satisfies the limit condition

\[(12) \quad \lim_{x \to \infty} \frac{g(x + t)}{g(x)} = 1 \quad \text{for each fixed } t.
\]

Let \(K\) be the class of all slowly varying functions.

For our purposes, \(K\) serves as an analogue to \(B\) and motivates much of our work.

**Theorem 3.** Let \(g\) be a slowly varying function. Then \(g\) satisfies (12) uniformly for \(t\) in bounded intervals, and there exist functions \(c(x)\) and \(\epsilon(x), \epsilon\)
continuous, \( \lim_{x \to \infty} c(x) = c \in (0, \infty) \), \( \lim_{x \to \infty} \epsilon(x) = 0 \) such that

\[
g(x) = c(x) \exp \int_0^x \epsilon(s) \, ds.
\]

Karamata [3] proved this theorem for \( g \) continuous. More recent work has weakened the hypothesis from continuity to measurability (see, for example, [1], [2], and [4]).

Relation between \( K \) and \( B_u \). There is a similarity between the class of uniformly \( B \)-slowly varying functions and the class \( K \), for a function \( \varphi \in B_u \) that is bounded away from zero on an appropriately chosen sequence is slowly varying:

**Theorem 4.** Let \( \varphi \in B_u \). If there is a sequence \( \{x_n\} \to \infty \), and constants \( m, M, \delta \) greater than zero such that

(i) \( m < x_{n+1} - x_n < M \) \( (n = 1, 2, \ldots) \),

(ii) \( \varphi(x_n) > \delta \) \( (n = 1, 2, \ldots) \)

then \( \varphi \in K \).

**Proof.** For each \( n = 1, 2, \ldots \), define the function \( p_n(x) \) for \( x \in [x_n, x_{n+1}] \) by

\[
p_n(x) = \frac{1}{2} \left( 1 + \sin \left( \frac{\pi}{2} \frac{2x - x_{n+1} - x_n}{x_{n+1} - x_n} \right) \right).
\]

Then \( 0 \leq p_n(x) \leq 1 \), \( p_n(x_n) = 0 \) and \( p_n(x_{n+1}) = 1 \). Moreover, the functions \( p_n \) are continuously differentiable,

\[
0 \leq p'_n(x) \leq \frac{\pi}{2(x_{n+1} - x_n)} \leq \frac{\pi}{2m}, \quad \text{and} \quad p'_n(x_n) = p'_n(x_{n+1}) = 0.
\]

Set \( f = \log \varphi \) and

\[
f_1(x) = f(x_n) + [f(x_{n+1}) - f(x_n)]p_n(x) \quad (x_n \leq x \leq x_{n+1}).
\]

The function \( f_1 \) is defined for all \( x \geq x_1 \), it is continuously differentiable, and satisfies the estimates

\[
|f'(x)| \leq (\pi/m)|f(x_{n+1}) - f(x_n)|
\]

and

\[
|f_1(x) - f(x)| \leq |f(x) - f(x_n)| + |f(x_{n+1}) - f(x_n)|
\]

where \( x_n \leq x \leq x_{n+1} \). Thus \( f_1 \) and \( f - f_1 \) tend to zero as \( x \) tends to infinity provided

\[
\lim_{x \to \infty} |f(x) - f(x_n)| = 0 \quad (x_n \leq x \leq x_{n+1}).
\]

For \( x_n \leq x \leq x_{n+1} \), there is a \( \epsilon \in (0, M/8] \) such that \( x = x_n + \epsilon \varphi(x_n) \). Then
which gives (16). Choose $\varepsilon(x)$ between zero and $x$, so that $\varepsilon$ is continuous, $\varepsilon(x_1) = f_1(x_1)$, and $\int_0^{x_1} \varepsilon(s) \, ds = f_1(x_1)$. Set

\begin{equation}
(18) \quad \varepsilon(x) = f_1'(x) \quad (x \geq x_1), \quad c(x) = \varphi(x) \exp\left(-\int_0^x \varepsilon(s) \, ds\right).
\end{equation}

Then for $x \geq x_1$, $c(x) = \exp[f(x) - f_1(x)] \to 1$, and $\varphi$ satisfies (13). It is a simple matter to verify that such functions are elements of $K$.

An integral representation. Motivating examples for the class of $B$-slowly varying functions are functions such as $x^p$ ($p < 1$) and $e^{-x}$. The derivatives of these functions tend to zero. This does not hold in general, as with $\varphi = x^{1/4} \sin x$, $x > 1$.

Here $\varphi \in B_u$, but $\lim \sup_{x \to \infty} \varphi'(x) \neq 0$.

But these examples suggest an analogue for the class $B$ of Karamata's representation (13).

**Theorem 5.** Let $\varphi \in B_u$. Then there are functions $c(x)$ and $\varepsilon(x)$, $\varepsilon$ continuous, $0 < c = \lim_{x \to \infty} c(x) < \infty$, and $\lim_{x \to \infty} \varepsilon(x) = 0$, such that

\begin{equation}
(20) \quad \varphi(x) = c(x) \int_0^x \varepsilon(s) \, ds.
\end{equation}

Conversely, if a positive, measurable function $\varphi$ has the representation (20) with $\varepsilon$ continuous, tending to zero and $c(x)$ tending to a positive limit, then $\varphi \in B_u$.

In particular, any $\varphi \in B_u$ satisfies (2).

The form of (20) was conjectured by Daniel Shea.

As an example, the function $\varphi$ given by (19) satisfies (20) with $c(x) = 1 + x^{-1/4} \sin x$, $\varepsilon(x) = \frac{1}{2} x^{-1/2}$.

We require some additional machinery before proving the theorem. Define inductively at $x$ a sequence $(x_n)$ by

\begin{equation}
(21) \quad x_0 = x, \quad x_n = x_{n-1} + \varphi(x_{n-1}).
\end{equation}

For $\varphi \in B_u$, this sequence virtually characterizes the behavior of $\varphi$ provided the $x_n$ become infinite.

**Lemma.** Let $\varphi \in B_u$. Then there is an $\bar{x}$ such that for any $x \geq \bar{x}$ and the sequence $(x_n)$ defined for $x$ by (21), $\lim_{n \to \infty} x_n = \infty$.

**Proof.** Choose $\bar{x}$ so that for $x \geq \bar{x}$, $t \in [-1, 1]$.

\begin{equation}
(22) \quad \varphi(x + t\varphi(x)) \geq \frac{1}{2} \varphi(x).
\end{equation}

Suppose the Lemma is false. Then there is an $x \geq \bar{x}$ with $(x_n)$ given by (21) such that $\lim_{n \to \infty} x_n = p < \infty$. This limit exists, of course, since the sequence $(x_n)$ increases monotonically. Now, $x_n = x + \sum_{k=1}^n \varphi(x_{k-1})$. The series
∑ \varphi(x_k) converges and, in particular, \lim_{n \to \infty} \varphi(x_n) = 0. Now, \( p \gg \chi \), so by (22),

\begin{equation}
\varphi(y) \geq \frac{1}{2} \varphi(p) \quad (p - \varphi(p) \leq y \leq p + \varphi(p)).
\end{equation}

Thus

\begin{equation}
\lim_{n \to \infty} \varphi(x_n) \geq \liminf_{y \to p} \varphi(y) \geq \frac{1}{2} \varphi(p),
\end{equation}

which contradicts the positivity of \( \varphi \).

**Proof of Theorem 5.** Let \( \chi \) be as in the Lemma and define \( \{x_n\} \) by (21) for some \( x_0 \gg \chi \). Then \( \lim_{n \to \infty} x_n = \infty \). Set

\begin{equation}
p_n(x) = \frac{\varphi(x_n)}{2} \left(1 + \sin \left[ \frac{\pi}{2\varphi(x_n)} \left(2x - 2x_n - \varphi(x_n)\right)\right]\right)
\end{equation}

\((x_n \leq x \leq x_{n+1})\).

\( p_n \) is continuously differentiable on \([x_n, x_{n+1}]\).

\begin{align*}
0 \leq p_n(x) &\leq \varphi(x_n), \quad p_n(x_n) = 0, \quad p_n(x_{n+1}) = \varphi(x_n), \\
0 \leq p'_n(x) &\leq \pi/2, \quad \text{and} \quad p'_n(x_{n+1}) = 0.
\end{align*}

Set

\begin{equation}
f(x) = \varphi(x_n) + p_n(x) \left[\frac{\varphi(x_{n+1}) - \varphi(x_n)}{\varphi(x_n)}\right] \quad (x_n \leq x \leq x_{n+1}).
\end{equation}

Then \( f \) is continuously differentiable, and

\begin{equation}
|f'(x)| \leq \pi/2 \left|\varphi(x_n + \varphi(x_n))/\varphi(x_n) - 1\right| \to 0 \quad \text{as} \quad x \to \infty.
\end{equation}

Finally, for \( x \in [x_n, x_{n+1}] \), there is a \( t \in [0,1] \) such that \( x = x_n + t\varphi(x_n) \). Thus

\begin{equation}
\frac{\varphi(x_n)}{\varphi(x_n + t\varphi(x_n))} \left[1 - \left|\frac{\varphi(x_n + \varphi(x_n))/\varphi(x_n) - 1}{\varphi(x)}\right|\right] \leq \frac{f(x)}{\varphi(x)}
\end{equation}

\begin{equation}
\leq \frac{\varphi(x_n)}{\varphi(x_n + t\varphi(x_n))} \left[1 + \left|\frac{\varphi(x_n + \varphi(x_n))/\varphi(x_n) - 1}{\varphi(x)}\right|\right].
\end{equation}

Hence, \( \lim_{x \to \infty} f(x)/\varphi(x) = 1 \). Define \( \epsilon(x) \) between zero and \( x_0 \) so that \( \epsilon \) is continuous, \( \epsilon(x_0) = f'(x_0) \), and \( \int_0^{x_0} \epsilon(s) \, ds = f(x_0) \). For \( x \gg x_0 \), set \( \epsilon(x) = f'(x) \). Assume further that \( \epsilon(x) > 0 \) for \( x \in [0, x_0] \). Set \( c(x) = \varphi(x)/\int_0^x \epsilon(s) \, ds \). Then (20) holds, \( \epsilon \) is continuous, and \( \lim_{x \to \infty} \epsilon(x) = 0 \). We need only show that \( c(x) \) tends to a finite positive limit. But for \( x \gg x_0 \), \( c(x) = \varphi(x)/f(x) \), which tends to one. The converse is easily verified.

**Remarks.** 1. Measurability of a slowly varying function \( g \) is sufficient to assure that the limit condition in (12) holds uniformly for \( t \) in finite intervals. An open problem at present is whether or not the hypothesis for Theorem 2 can be similarly weakened.
The contradiction in the proof of Theorem 2 was obtained by constructing sets $W'_n$ of measure bounded below and all contained in a finite interval.

Since

$$\mathfrak{m}(W'_n) \geq \inf_n (\varphi(y_n)/\varphi(x_n)),$$

and $W'_n \subset (0, 1 + \sup_n (\varphi(y_n)/\varphi(x_n)))$,

it is impossible for $\varphi \in B$ to have constants $m, M, \delta > 0$ and sequences $(t_n) \subset (0, 1), (x_n) \to \infty$, such that $|\varphi(y_n)/\varphi(x_n) - 1| \geq \delta$ and $m \leq \varphi(y_n)/\varphi(x_n) \leq M$, where $y_n = x_n + t_n \varphi(x_n)$. For continuous functions, the second of these inequalities follows from the first. Measurable functions are not as simple, however. Consider, for example,

$$f(x, t) = \begin{cases} 1/x, & x \geq t > 0, \\ 0, & 0 < x < t. \end{cases}$$

For each fixed $t > 0$, $\lim_{x \to 0} f(x, t) = 0$, but for any $\epsilon > 0$, if we choose sequences $(x_n) \to 0$, $(t_n)$ so that $f(x_n, t_n) \geq \epsilon$ ($n = 1, 2, 3, \ldots$), then $\lim_{n \to \infty} f(x_n, t_n) = \infty$.

Let $\delta_n = \varphi(y_n)/\varphi(x_n)$, and suppose $\lim_{n \to \infty} \delta_n = \infty$. What can be said about the sequence $(\delta_n)$ relative to $\varphi$?

Fix $t \in [1, 2]$ and let $k$ be an integer, $k \geq 4$. Let $\epsilon \in (0, 2^{1/k} - 1)$ be given, and set

$$V_n(t) = \{\mu \in (t_n, t_n + t) : |\varphi(x_n + \mu x(x_n))/\varphi(x_n) - 1| < \epsilon\},$$

$$Q_n(t) = \{\lambda \in [0, 2^t] : |\varphi(y_n + \lambda \varphi(y_n))/\varphi(y_n) - 1| > 1/4\}.$$

Then

$$\lim_{n \to \infty} \mathfrak{m}(V_n(t)) = t, \quad \lim_{n \to \infty} \mathfrak{m}(Q_n(t)) = 0.$$

There is an $\hat{x}$ such that for $x \geq \hat{x}$,

$$|\varphi(x + \epsilon \varphi(x))/\varphi(x) - 1| < \epsilon.$$

Suppose that $\delta_n \in [k - 1, k]$ for some $x_n \geq x$. For $\lambda_1 \in V_n(t)$, define $\lambda_j$ by

$$\lambda_j = \lambda_{j-1} + t \varphi(x_n + \lambda_{j-1} \varphi(x_n))/\varphi(x_n) \quad (2 \leq j \leq k).$$

Then

$$\varphi(y_n + ((\lambda_j - t_n)/\delta_n) \varphi(y_n))/\varphi(y_n) = \varphi(x_n + \lambda_j \varphi(x_n))/\varphi(x_n)$$

$$\leq (1 + \epsilon)^{j-1} \varphi(x_n) \leq (1 + \epsilon)^{k-1} \delta_n^{-1} \leq 2/(k - 1).$$

Set $Q'_n = \{(\lambda_j - t_n)/\delta_n : \lambda_j \in V_n(t), \lambda_j \text{ defined by } (32)\}$. Then

$$\mathfrak{m}(Q'_n) \geq \inf_{\delta_n} \frac{(\lambda_j - \lambda_{j-1})(k - 1)\mathfrak{m}(V_n)}{\delta_n} \geq \frac{t(k - 1)(1 - \epsilon)}{k} \mathfrak{m}(V_n(t)).$$

But $Q'_n \subset Q_n(t)$ for all $n = 1, 2, \ldots$, which contradicts (30).
Thus, given \( \{t_n\} \subset (0, 1) \), \( \{x_n\} \to \infty \) such that \( \delta_n = \varphi(x_n + t_n \varphi(x_n))/\varphi(x_n) \to \infty \), let \( k(n) \) be the least integer greater than \( \delta_n - \lceil -\delta_n \rceil \). We conclude that

\[
\left\{ t \in [1, 2] : \left| \frac{q(x + t \varphi(x))}{q(x)} - 1 \right| < 2^{1/k(n)} - 1 \right\} \text{ for all } x \geq x_n, = \emptyset,
\]

and we have a uniform bound for the rate of convergence in the \( B \)-slowly varying limit. A similar argument is applicable when \( \lim_{n \to \infty} \delta_n = 0 \).

2. Theorem 4 considers uniformly \( B \)-slowly varying functions bounded away from zero on appropriate sequences. Suppose we add a positive constant to an element in \( B_\nu \). What can we conclude about this translate?

**Theorem 6.** Let \( \varphi \in B_\nu \), \( T(x) = \varphi(x) + \epsilon \), \( \epsilon > 0 \). Then \( T \in B_\nu \cap K \).

**Proof.** Fix \( t \). Then

\[
\lim_{x \to \infty} \frac{T(x - t \varphi(x))}{T(x)} = \lim_{x \to \infty} \frac{\epsilon + \varphi(x + t \varphi(x))}{\epsilon + \varphi(x)} = 1,
\]

and this limit holds uniformly for \( t \) in finite intervals. In the proof of Theorem 4, set \( x_n = n \), and \( f(x) = \log[T(x - \epsilon(x - n)/T(n))] \) \( (n \leq x < n + 1) \). Define \( f \) by (15) and let \( t = (x - n)/T(n) \) for \( x \in [n, n + 1] \). Then \( 0 \leq t \leq 1/T(n) \leq 1/\epsilon \), so for \( x \leq [n, n + 1] \),

\[
|f(x) - f(n)| = \left| \log \frac{T(x - \epsilon t)}{T(n)} \right| = \left| \log \frac{T(n - \epsilon t + \epsilon T(n))}{T(n)} \right| \to 0.
\]

So (16) holds, and \( T(x - \epsilon(x - n)/T(n)) \in K \), with

\[
T(x - \epsilon \frac{x - n}{T(n)}) = c(x) \exp \int_1^x \epsilon(s) \, ds.
\]

\[
\lim_{x \to \infty} \frac{T(x - \epsilon(x - n)/T(n) + t)}{T(x - \epsilon(x - n)/T(n))} = 1.
\]

and this limit holds uniformly for \( t \in [0, 1] \), so that for \( t = \epsilon(x - n)/T(n) \),

\[
c_1(x) = \frac{T(x)}{T(x - \epsilon(x - n)/T(n))} \to 1 \text{ as } x \to \infty.
\]

Therefore,

\[
T(x) = c_1(x)c(x) \exp \int_1^x \epsilon(s) \, ds,
\]

and so \( T \in K \).

Let \( \{x_n\} \) be a sequence tending to infinity, and set

\[
\{x_{n_1}\} = \{x_n : \varphi(x_n) \leq 1\}, \quad \{x_{n_2}\} = \{x_n : \varphi(x_n) > 1\}.
\]

Fix \( t \), and set

\[
\lambda_{n_1} = tT(x_{n_1}), \quad \mu_{n_2} = t + t\epsilon/\varphi(x_{n_2}).
\]
Then \( \lambda_{n_\alpha}, \mu_{n_\beta} \in [t, t + t\epsilon] \). Let \( \eta > 0 \) be given. Then there are integers \( N_1 \) and \( N_2 \) such that

\[
\left| \frac{T(x + \lambda)}{T(x)} - 1 \right| < \eta \quad (x \geq x_{N_1}; \lambda \in [t, t + \epsilon]),
\]

(41)

\[
\left| \frac{\varphi(x + \mu \varphi(x))}{\varphi(x)} - 1 \right| < \eta \quad (x \geq x_{N_2}; \mu \in [t, t + t\epsilon]),
\]

since \( T \in K \) and \( \varphi \in B_u \). Let \( N = \max\{N_1, N_2\} \). Then, for \( n \geq N \), \( \varphi(x_n) \geq 1 \),

\[
\left| \frac{T(x_n + tT(x_n))}{T(x_n)} - 1 \right| = \left| \frac{\varphi(x_n + \mu_n \varphi(x_n)) + \epsilon}{\varphi(x_n) + \epsilon} - 1 \right| < \eta \varphi(x_n) / (\varphi(x_n) + \epsilon) < \eta.
\]

While if \( \varphi(x_n) < 1 \),

\[
\left| \frac{T(x_n + tT(x_n))}{T(x_n)} - 1 \right| = \left| \frac{T(x_n + \lambda_n)}{T(x_n)} - 1 \right| < \eta.
\]

The bound in (41) holds uniformly for \( t \) in finite intervals, so \( T \in B_u \), which completes the proof.

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**Bibliography**


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