ON $L^1$ CONVERGENCE OF CERTAIN COSINE SUMS

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Abstract. Rees and Stanojević introduced a new class of modified cosine sums $(g_n(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a(k) + \sum_{k=1}^{n} \Delta a(j) \cos kx)$ and found a necessary and sufficient condition for integrability of these modified cosine sums. Here we show that to every classical cosine series $f$ with coefficients of bounded variation, a Rees-Stanojević cosine sum $g_n$ can be associated such that $g_n$ converges to $f$ pointwise, and a necessary and sufficient condition for $L^1$ convergence of $g_n$ to $f$ is given. As a corollary to that result we have a generalization of the classical result of this kind. Examples are given using the well-known integrability conditions.

Theorem A gives a necessary and sufficient condition for a sine series with coefficients of bounded variation and converging to zero to be the Fourier series of its sum, or equivalently, for its sum to be integrable. Theorem B shows that if such a series is a Fourier series then its convergence is "good", that is, convergence in the $L^1$ metric.

**Theorem A** [1]. Let $f(x) = \sum_{n=1}^{\infty} b(n) \sin nx$ where $\Delta b(n) = b(n) - b(n + 1)$ and $\lim_{n \to \infty} b(n) = 0$. Then $f \in L^1[0,\pi]$ if and only if $\sum_{n=1}^{\infty} |\Delta b(n)| \log n < \infty$.

**Theorem B** [1]. Let $f(x)$ be as in Theorem A. If $f \in L^1[0,\pi]$ then $\sum_{k=1}^{n} b(k) \sin kx$ converges to $f$ in the $L^1$ metric.

There is no known analogue of Theorem A for the cosine series. Theorems C and D only give sufficient conditions for the cosine series to be the Fourier series of its sum.

In what follows we will denote by $C$ the cosine series

$$\frac{1}{2}a(0) + \sum_{n=1}^{\infty} a(n) \cos nx$$

where $\lim_{n \to \infty} a(n) = 0$ and $\sum_{n=1}^{\infty} |\Delta a(n)| < \infty$. Partial sums of $C$ will be denoted by $S_n(x)$, and $f(x) = \lim_{n \to \infty} S_n(x)$.

**Theorem C** [1]. If $\sum_{n=1}^{\infty} |\Delta a(n)| \log n < \infty$, then $f \in L^1[0,\pi]$ or, equivalently, $C$ is the Fourier series of $f$.

**Theorem D** [1]. If $\sum_{n=1}^{\infty} |\Delta^2 a(n)|(n + 1) < \infty$, then $f \in L^1[0,\pi]$ or, equivalently, $C$ is the Fourier series of $f$.

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1 Portions of these results appear in a doctoral thesis of John W. Garrett at the University of Missouri-Rolla in 1974.

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Theorem E is related to Theorem B. It shows that the classical cosine series is not as "well behaved" as the classical sine series.

**Theorem E** [1]. If $\sum_{n=1}^{\infty} |\Delta^2 a(n)|(n + 1) < \infty$, then $S_n$ converges to $f$ in the $L^1$ metric if and only if $\lim_{n \to \infty} a(n) \log n = 0$.

Rees and Stanojević introduced a new type of cosine sum and obtained a necessary and sufficient condition for integrability of its limit.

**Theorem F** [2]. Let 
\[
g^*(x) = \sum_{k=0}^{n} \frac{a(k)}{2} + \sum_{j=k}^{n} a(j) \cos kx.
\]
where $\lim_{n \to \infty} a(n) = 0$ and $\Delta a(n) \geq 0$. Then 
(i) $g^*(x) = \lim_{n \to \infty} g^*_n(x)$ exists for $x \in (0, \pi]$, and
(ii) $g^* \in L^1[0,\pi]$ if and only if $\sum_{n=1}^{\infty} a(n) < \infty$.

This paper proves an analogue of Theorem B for this type of cosine sum. Indeed, these modified cosine sums approximate their limit "better" than the classical cosine series since they converge in the $L^1$ metric to their limit when the classical cosine series may not.

**Lemma 1.** Let 
\[
g_n^*(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a(k) + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a(j) \cos kx.
\]
Then $\lim_{n \to \infty} g_n(x) = f(x)$, for $x \in (0, \pi]$.

It will be shown in the proof of this lemma that 
\[
g_n(x) = S_n(x) - a(n + 1) D_n(x).
\]
We prefer the form given in the lemma, however, since it emphasizes better its use in [2].

**Proof.** Denoting the Dirichlet kernel by $D_n(x)$ we get 
\[
\lim_{n \to \infty} g_n(x) = \lim_{n \to \infty} \left[ \frac{1}{2} \sum_{k=0}^{n} \Delta a(k) + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a(j) \cos kx \right]
\]
\[
= \lim_{n \to \infty} \left[ \frac{a(0)}{2} + \sum_{k=1}^{n} a(k) \cos kx - a(n + 1) D_n(x) \right]
\]
\[
= \lim_{n \to \infty} [S_n(x) - a(n + 1) D_n(x)] = f(x),
\]
$x \in (0, \pi]$ since $\lim_{n \to \infty} S_n(x) = f(x)$ and $\lim_{n \to \infty} a(n + 1) D_n(x) = 0$, $x \in (0, \pi]$.

**Theorem 1.** Let $g_n$ be as defined in Lemma 1. Then $g_n$ converges to $f$ in the $L^1$ metric if and only if given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that 
\[
\int_{0}^{\delta} |\sum_{k=n+1}^{\infty} \Delta a(k) D_k(x)| < \varepsilon \text{ for all } n \geq 0.
\]

**Proof.** For the "if" part let $\varepsilon > 0$. Then there exists $\delta > 0$ such that
\[
\begin{align*}
\int_{0}^{\pi} |f - g_n| &= \int_{0}^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a(k) D_k(x) \right| \\
&= \int_{0}^{\delta} \left| \sum_{k=n+1}^{\infty} \Delta a(k) D_k(x) \right| + \int_{\delta}^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a(k) D_k(x) \right| \\
&< \frac{\varepsilon}{2} + \sum_{k=n+1}^{\infty} |\Delta a(k)| \int_{0}^{\pi} |D_k(x)| \\
&\leq \frac{\varepsilon}{2} + \sum_{k=n+1}^{\infty} |\Delta a(k)| \int_{0}^{\pi} \csc \frac{1}{2}x \\
&= \frac{\varepsilon}{2} + \sum_{k=n+1}^{\infty} |\Delta a(k)| \left[-2 \log |\csc \delta/2 - \cot \delta/2| \right] < \varepsilon
\end{align*}
\]

for sufficiently large \( n \) since \( \sum_{k=0}^{\infty} |\Delta a(k)| < \infty \).

For the "only if" part, let \( \varepsilon > 0 \). Then there exists an integer \( M \) such that
\[
\int_{0}^{\pi} |f(x) - g_n(x)| < \varepsilon/2 \text{ if } n \geq M.
\]
That is, \( \int_{0}^{\pi} |\sum_{k=n}^{\infty} \Delta a(k) D_k(x)| < \varepsilon/2 \) if \( n \geq M \). Now if \( \sum_{k=0}^{M} |\Delta a(k)| = 0 \), then for \( n > M \),
\[
\int_{0}^{\pi} \left| \sum_{k=n}^{\infty} \Delta a(k) D_k(x) \right| < \varepsilon/2 < \varepsilon \text{ and, for } 0 \leq n \leq M,
\]
\[
\int_{0}^{\pi} \left| \sum_{k=n}^{\infty} \Delta a(k) D_k(x) \right| < \int_{0}^{\pi} \left| \sum_{k=n}^{M} \Delta a(k) D_k(x) \right| < \varepsilon/2 < \varepsilon.
\]
If \( \sum_{k=0}^{M} |\Delta a(k)| \neq 0 \), let \( \delta = \varepsilon/2M \sum_{k=0}^{M} |\Delta a(k)| \). For \( n \geq M \),
\[
\int_{0}^{\delta} \left| \sum_{k=n}^{\infty} \Delta a(k) D_k(x) \right| \leq \int_{0}^{\pi} \left| \sum_{k=n}^{\infty} \Delta a(k) D_k(x) \right| < \varepsilon/2 < \varepsilon.
\]
For \( 0 \leq n < M \),
\[
\begin{align*}
\int_{0}^{\delta} \left| \sum_{k=n}^{\infty} \Delta a(k) D_k(x) \right| &\leq \int_{0}^{\delta} \left| \sum_{k=n}^{M-1} \Delta a(k) D_k(x) \right| + \int_{0}^{\delta} \left| \sum_{k=M}^{\infty} \Delta a(k) D_k(x) \right| \\
&\leq \int_{0}^{\delta} \left| \sum_{k=n}^{M-1} k |\Delta a(k)| \right| + \int_{0}^{\pi} \left| \sum_{k=M}^{\infty} \Delta a(k) D_k(x) \right| \\
&< \delta \sum_{k=0}^{M-1} k |\Delta a(k)| + \frac{\varepsilon}{2} \\
&\leq \delta M \sum_{k=0}^{M-1} |\Delta a(k)| + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{align*}
\]
So given \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( \int_{0}^{\delta} |\sum_{k=n}^{\infty} \Delta a(k) D_k(x)| < \varepsilon \) for all \( n \geq 0 \).

If \( \lim_{n \to \infty} \int_{0}^{\pi} |f(x) - g_n(x)| = 0 \), it is clear that \( f \in L^1[0, \pi] \).
\[
\int_{0}^{\pi} |f(x)| \leq \int_{0}^{\pi} |f(x) - g_n(x)| + \int_{0}^{\pi} |g_n(x)| < \infty
\]
since \( g_n(x) \) is a finite cosine sum.
\[ f(x) = \sum_{k=n}^{\infty} \Delta a(k) D_k(x) \] \( f(x) \) for all \( n \geq 0 \), then \( S_n \) converges to \( f \) in the \( L^1 \) metric if and only if \( \lim_{n \to \infty} a(n) \log n = 0 \).

**Proof.** Using \( g_n \) as defined in Lemma 1, we get

\[
\int_0^\pi |f(x) - S_n(x)| = \int_0^\pi |f(x) - g_n(x) + g_n(x) - S_n(x)|
\]

\[
\leq \int_0^\pi |f(x) - g_n(x)| + \int_0^\pi |g_n(x) - S_n(x)|
\]

\[
= \int_0^\pi |f(x) - g_n(x)| + \int_0^\pi |a(n + 1) D_n(x)|.
\]

Also

\[
\int_0^\pi |a(n + 1) D_n(x)| = \int_0^\pi |g_n(x) - S_n(x)|
\]

\[
\leq \int_0^\pi |f(x) - S_n(x)| + \int_0^\pi |f(x) - g_n(x)|.
\]

Since \( \int_0^\pi |a(n + 1) D_n(x)| \) behaves like \( a(n + 1) \log n \) for large values of \( n \), and \( \lim_{n \to \infty} \int_0^\pi |f(x) - g_n(x)| = 0 \), the corollary is proved.

The following examples show that known sufficient conditions for integrability of the limit of a cosine series are also sufficient for the \( L^1 \) convergence of \( g_n \) to that limit, since they imply the necessary and sufficient condition from Theorem 1.

**Example 1.** Let \( \sum_{n=1}^{\infty} |A^2 a(n)|(n + 1) < \infty \). Then \( g_n \) converges to \( f \) in the \( L^1 \) metric space. Denoting the Fejér kernel by \( F_n(x) \), we get

\[
\int_0^\pi |f(x) - g_n(x)| = \int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta a(k) D_k(x) \right|
\]

\[
= \int_0^\pi \left| \sum_{k=n+1}^{\infty} (k + 1) \Delta^2 a(k) F_k(x) - (n + 1) \Delta a(n) F_n(x) \right|
\]

\[
\leq \sum_{k=n+1}^{\infty} (k + 1) |\Delta^2 a(k)| \int_0^\pi F_k(x) + (n + 1) |\Delta a(n)| \int_0^\pi F_n(x)
\]

\[
\leq \pi \sum_{k=n+1}^{\infty} (k + 1) |\Delta^2 a(k)|
\]

since \( \int_0^\pi F_k(x) = \pi/2 \) and

\[
(n + 1) |\Delta a(n)| = \sum_{k=n}^{\infty} (n + 1) \left( \left| \Delta a(k) \right| - |\Delta a(k + 1)| \right)
\]

\[
\leq \sum_{k=n}^{\infty} (n + 1) |\Delta^2 a(k)| \leq \sum_{k=n}^{\infty} (k + 1) |\Delta^2 a(k)|.
\]

Since \( \sum_{n=1}^{\infty} (n + 1) |\Delta^2 a(n)| < \infty \), then \( \lim_{n \to \infty} \int_0^\pi |f(x) - g_n(x)| = 0 \).

**Example 2.** Let \( \sum_{k=n}^{\infty} |\Delta a(k)| \log k < \infty \). Then \( g_n \) converges to \( f \) in the \( L^1 \) metric space, for
\[ \int_0^\pi |f(x) - g_n(x)| = \int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta a(k) D_k(x) \right| \]
\[ \leq \sum_{k=n+1}^{\infty} |\Delta a(k)| \int_0^\pi |D_k(x)|. \]

Since \( \int_0^\pi |D_k(x)| \) behaves like \( \log k \) for large \( k \), and \( \sum_{k=1}^{\infty} |\Delta a(k)| \log k < \infty \), we get \( \lim_{n \to \infty} \int_0^\pi |f(x) - g_n(x)| = 0 \). As a corollary of this example we have the well-known Theorem E.

Theorems C and D can be combined as in the following lemma.

**Lemma 2.** Let \( a(n) = b(n) + c(n) \) where \( \sum_{n=1}^{\infty} |\Delta b(n)| \log n < \infty \), \( \sum_{n=1}^{\infty} |\Delta^2 c(n)|(n + 1) < \infty \), and \( \lim_{n \to \infty} b(n) = \lim_{n \to \infty} c(n) = 0 \). Then \( f \in L^1[0, \pi] \).

It is interesting to note that in Lemma 2 we may have
\[ \sum_{n=1}^{\infty} |\Delta a(n)| \log n = \sum_{n=1}^{\infty} |\Delta^2 a(n)|(n + 1) = \infty. \]

**Example 3.** Let \( f(x) \) be as in Lemma 2. Then \( g_n \) converges to \( f \) in the \( L^1 \) metric. This follows from Examples 1 and 2, writing \( a(n) = b(n) + c(n) \).

Stanojević combined Theorems C and D in a different way.

**Theorem G** [3]. Let \( a(n) = a(n)\beta(n) \) where \( \sum_{n=1}^{\infty} |\Delta a(n)| < \infty \), \( \sum_{n=1}^{\infty} |\Delta^2 \beta(n)|(n + 1) < \infty \), \( |\beta(n)| \leq M \), and \( \sum_{n=1}^{\infty} |\beta(n)\Delta a(n)| \log(n) < \infty \). Then \( f \in L^1[0, \pi] \).

**Example 4.** Let \( f(x) \) be as in Theorem G. Then \( g_n \) converges to \( f \) in the \( L^1 \) metric. We get
\[ \int_0^\pi |f(x) - g_n(x)| \leq M \sum_{k=n}^{\infty} |\beta(k)\Delta a(k)| \log k \]
\[ + N \sum_{k=n}^{\infty} (k + 1)(|\alpha(k + 1)\Delta^2 \beta(k)| + |\Delta a(k + 1)\Delta \beta(k + 1)|). \]

Since both series converge, we have \( \lim_{n \to \infty} \int_0^\pi |f(x) - g_n(x)| = 0 \).

**References**