AN ABELIAN ERGODIC THEOREM FOR SEMIGROUPS IN $L_p$ SPACE

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Abstract. The purpose of this paper is to prove individual and dominated ergodic theorems for Abel means of semigroups of positive $L_p$ contractions, $1 < p < \infty$.

1. Introduction. Let $(X, \Sigma, \mu)$ be a $\sigma$-finite measure space and $L_p(\mu) = L_p(X, \Sigma, \mu)$, $1 \leq p \leq \infty$, the usual Banach spaces of complex-valued functions. Let $\{T(t): t \geq 0\}$ be a strongly measurable semigroup of positive $L_p(\mu)$ contractions for some $1 < p < \infty$. This means that (i) $\|T(t)\|_p \leq 1$, $t \geq 0$; (ii) $0 \leq f \in L_p(\mu) \Rightarrow T(t)f \geq 0$; (iii) $T(s + t) = T(s)T(t)$, $s, t \geq 0$; (iv) $f \in L_p(\mu) \Rightarrow T(\cdot)f$ is measurable with respect to Lebesgue measure on the interval $[0, \infty)$. For $\lambda > 0$ we set

$$R_\lambda f(x) = \int_0^\infty e^{-\lambda t} T(t)f(x)\,dt$$

for $f \in L_p(\mu)$. In this paper we prove that

$$\int \left( \limsup_{\lambda \to 0+} \lambda R_\lambda f(x) \right)^p d\mu \leq (p/p - 1)^p \int |f|^p d\mu$$

and $\lim_{\lambda \to 0+} \lambda R_\lambda f(x)$ exists and is finite for a.e. $x \in X$. Before proceeding we justify the definition of $R_\lambda f(x)$. By Theorem III.11.17 in [3], given $f \in L_p(\mu)$ and $\lambda > 0$ the strong measurability of $\{T(t)\}$ guarantees the existence of a function $g_\lambda(t, x)$ on the product space $[0, \infty) \times X$, measurable with respect to the usual product $\sigma$-field, which is uniquely determined up to a set of measure zero in this space by the conditions (i) $g_\lambda(t, \cdot) = e^{-\lambda t} T(t)f$ for a.e. $t$, (ii) for a.e. $x$, $g_\lambda(\cdot, x)$ is integrable over $[0, \infty)$ and $\int_0^\infty g_\lambda(t, x)\,dt$ as a function of $x$ is equal a.e. to $\int_0^\infty e^{-\lambda t} T(t) f(\cdot)\,dt$ defined as the $L_p$ limit of Riemann sums. The set on which

$$\int_0^\infty g_\lambda(t, x)\,dt \neq \int_0^\infty e^{-\lambda t} T(t)f(\cdot)\,dt$$

is independent of $\lambda > 0$. We define $R_\lambda f(x) = \int_0^\infty g_\lambda(t, x)\,dt$. This justifies the definition of $R_\lambda f(x)$.

In a recent paper of R. Sato [7] it was shown that if $f \in L_p(\mu)$ then $\|f^*\| \leq (p/(p - 1)) \|f\|$ and $\lim_{\lambda \to 0+} \lambda R_\lambda f(x)$ exists and is finite a.e. on $X$. The function $f^*$ is given by $f^* = \sup_{\lambda > 0} |\lambda R_\lambda f(x)|$. He also obtained a weak
estimate for $f^*$ in case $f \in L_1(\mu)$. Sato obtained these results assuming \( \{T(t)\} \)
to be a strongly measurable semigroup of (not necessarily positive) $L_1(\mu)$
contractions satisfying $\|T(t)f\|_\infty \leq \|f\|_\infty$ for all $f \in L_1(\mu) \cap L_\infty(\mu)$. In this
paper we obtain Sato’s $L_p$ results assuming $\{T(t)\}$ is a semigroup of positive
$L_p(\mu)$ contractions for some $1 < p < \infty$.

ACKNOWLEDGMENT. The author wishes to thank Professor M. A. Akcoglu
for making his manuscript [1] available to him prior to publication.

2. Preliminary results. Our purpose in this section is to establish the
dominated estimate $\|f^*\| \leq (p/(p - 1))\|f\|$ for a discrete semigroup. Let $T$
be a positive contraction of $L_p(\mu)$. Throughout this section we let
$R_\lambda f = \sum_0^\infty \lambda^n T^n f$, $f \in L_p(\mu)$, $0 < \lambda < 1$, and $f^* = \sup_0<\lambda<1|(1 - \lambda)R_\lambda f|$. We
say that $T$ admits of a dominated estimate with constant $c > 0$ if $\|f^*\|
\leq c\|f\|$, $f \in L_p(\mu)$.

1. Lemma. Let $T_n$, $n = 1, 2, \ldots$, and $T$ be positive contractions of $L_p(\mu)$ such
that each $T_n$ admits of a dominated estimate with constant $c$. If $\{T_n\}$ converges
strongly to $T$ then $T$ also admits of a dominated estimate with constant $c$.

Proof. The argument is analogous to that appearing in [5, p. 369]. Let
$A_1, \ldots, A_n$ be disjoint measurable sets and $k$ a positive integer. For any
$f \in L_p^+(\mu)$ and $0 < \lambda_j < 1$, $j = 1, 2, \ldots, n$, we have

$$\left\| \sum_{j=1}^n (1 - \lambda_j)X_{A_j}(f + \lambda_j T_i f + \cdots + \lambda_j^k T_i^k f) \right\| \leq c\|f\|$$

for $i = 1, 2, 3, \ldots$. Since $\{T_i\}$ converges strongly to $T$, the above estimate
holds with $T_i$ replaced by $T$. It follows that $\left\| \sum_j (1 - \lambda_j)X_{A_j}R_\lambda f \right\| \leq c\|f\|$. By
the monotone convergence theorem we get

$$\left\| \sup_{0<\lambda<1} (1 - \lambda)R_\lambda f \right\| \leq c\|f\|.$$

Since $(1 - \lambda)R_\lambda f$ depends continuously on $\lambda$, it follows that

$$\left\| \sup_{0<\lambda<1} (1 - \lambda)R_\lambda f \right\| \leq c\|f\|,$$

$f \in L_p^+(\mu)$. Clearly the estimate also holds for arbitrary $f \in L_p(\mu)$. Q.E.D.

2. Lemma. Let $(X, \Sigma, \mu)$ be a Lebesgue space and $T$ a positive invertible
isometry of $L_p(\mu)$. Then $T$ admits of a dominated estimate with constant
$p/(p - 1)$.

Proof. It is well known (see [5], [6]) that $T$ is induced by an invertible point
transformation and that, as a consequence of Linderholm's theorem [4, p. 71],
$T$ may be approximated in the strong operator topology by positive periodic
isometries. Hence by Lemma 1 it is sufficient to prove the lemma assuming $T$
is a positive periodic isometry. If $0 < f \in L_p(\mu)$ and $T$ has period $n$, then
$h = \sum_0^{n-1} T^n f$ is a positive fixed function for $T$, i.e. $Th = h$. As in [2] define a
measure $m$ on $\Sigma$ by $m(A) = \int_A h d\mu$, and an operator $P$ on $L_p(m)$ by
$P(f) = T(fh)/h$, $f \in L_p(m)$. By Lemma 3.1 in [2], $\|P\| \leq 1$, $\|P\|_\infty \leq 1$. License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Consequently $P$ admits of a dominated estimate with constant $p/(p - 1)$ by Theorem 2 in [7]. For $f \in L_p(\mu)$, we have $f/h \in L_p(m)$ and $P^n(f/h) = T^n(f)/h, n = 0, 1, 2, \ldots$. Hence

$$\int \sup \left(1 - \lambda \right) \sum_0^\infty \lambda^n T^n f \, d\mu = \int \sup \left(1 - \lambda \right) \sum_0^\infty \lambda^n P^n(f/h) \, d\mu \leq \left(p/(p - 1)\right)^n \int \left| f \right|^p \, d\mu.$$ 

Thus $T$ admits of a dominated estimate with constant $p/(p - 1)$. Q.E.D.

We now show that every positive contraction of $L_p(\mu)$ admits of a dominated estimate. We proceed as in [1]: the estimate is obtained first for positive contractions (matrices) operating on $l_p$, where $l_p$ is the $L_p$ space consisting of functions $r = (r_i) \in R^n$ whose norms are given by $\|r\|_p = \left[\sum_1^n |r_i|^p m_i\right]^{1/p}$, where the $m_i$'s are fixed positive numbers. Some of the details in the proofs of the following lemmas are omitted since the arguments are similar to those in [1].

3. Lemma. Let $T: l_p \rightarrow l_p$ be a positive contraction. Then $T$ admits of a dominated estimate with constant $p/(p - 1)$.

Proof. The operator $T$ is given by an $n \times n$ matrix $(T_{ij})$ whose entries $T_{ij}$ are nonnegative. By Lemma 1 it is enough to establish the lemma assuming each $T_{ij} > 0$. Clearly we may assume $\|T\| = 1$. Given these conditions on $T$ we construct as in [1] a space $(Z, \mathcal{B}, \nu)$ where $Z = \cup_i E_i$, $E_i$ a rectangle in $R_2$, $\mathcal{B}$ is the collection of two dimensional Borel subsets of $Z$, $\nu$ is the restriction of two dimensional Lebesgue measure to $\mathcal{B}$. The $E_i$'s satisfy $\nu(E_i) = m_i$. For a given $r = (r_i) \in l_p$ set $f = \sum_1^n r_i \chi_{E_i}$. There exists a positive invertible isometry $Q$ on $L_p(Z)$ such that for $i = 0, 1, 2, \ldots$

$$EQ^i f = \sum_1^n (T^i r) \chi_{E_i},$$

where $E$ is the conditional expectation operator on $L_p(Z)$ with respect to $\{E_i\}$. Setting $f^* = \sup_{0 < \lambda < 1} \left(1 - \lambda \right) \sum_0^\infty \lambda^i Q^i f$, we have

$$\|f^*\| \leq (p/(p - 1)) \|f\| = (p/(p - 1)) \|r\|$$

by Lemma 2. But $\sup_{0 < \lambda < 1} \left(1 - \lambda \right) \sum_0^\infty \lambda^i EQ^i f \leq Ef^*$ and

$$\sup_{0 < \lambda < 1} \left(1 - \lambda \right) \sum_0^\infty \lambda^i EQ^i f = \sup_{0 < \lambda < 1} \left(1 - \lambda \right) \sum_1^n \sum_0^\infty \lambda^i (T^i r) \chi_{E_i}$$

$$= \sum_1^n f^*_j \chi_{E_j}.$$ 

Thus $\|r^*\| = \|\sum f^*_j \chi_{E_j}\| \leq \|Ef^*\| \leq (p/(p - 1)) \|r\|$. Q.E.D.

4. Lemma. Let $T$ be a positive contraction of $L_p(\mu)$. Then $T$ admits of a dominated estimate with constant $p/(p - 1)$.

Proof. Suppose the theorem is false. Then there exists $f \in L_p^+(\mu), K \geq 1, 0 < \lambda_j < 1, j = 1, 2, \ldots, k$ such that
\[ \left\| \sup_j (1 - \lambda_j) \sum_{i=0}^{K} \lambda_j^i T^i f \right\| > \left( \frac{p}{p-1} \right) \|f\|. \]

By Lemmas 3.1 and 3.2 in [1] there exists a conditional expectation \( E \) on \( L_p(\mu) \) such that
\[ \left\| \sup_j (1 - \lambda_j) \sum_{i=0}^{K} \lambda_j^i (ET)^i Ef \right\| > \left( \frac{p}{p-1} \right) \|Ef\|. \]

Let \( \{E_1, \ldots, E_n\} \) be the partition of \( X \) corresponding to \( E \) and \( \{E_{i_1}, \ldots, E_{i_m}\} \) the atoms of \( \{E_{i_1}\} \) having finite positive measure. The subspace of \( L_p(\mu) \) of functions which are constant on these atoms can be identified with \( L_p \) and \( ET \) defines a positive contraction on this \( L_p \). Then the preceding inequality contradicts Lemma 3. Q.E.D.

3. Main results. Throughout this section we set
\[ R_\lambda f(x) = \lambda \int_0^\infty e^{-\lambda t} T(t) f(x) \, dt \]
and
\[ f^* = \sup_{0<\lambda<\infty} |\lambda R_\lambda f(x)|, \quad f \in L_p(\mu). \]

5. Lemma. For \( f \in L_p(\mu) \) we have \( f^* \in L_p(\mu) \) and
\[ \|f^*\| \leq \left( \frac{p}{p-1} \right) \|f\|. \]

Proof. As in [7, pp. 544-545], one can show there exists a sequence \( \{n_i\} \) such that for any rational \( \lambda > 0 \),
\[ \lambda R_\lambda f(x) = \lim_i (1 - e^{-\lambda/n_i}) \sum_{k=0}^\infty e^{-\lambda k/n_i} T(k/n_i) f(x) \quad \text{a.e.} \]
Setting
\[ f_i^*(x) = \sup_{0<\lambda<\infty} (1 - e^{-\lambda/n_i}) \sum_{k=0}^\infty e^{-\lambda k/n_i} T(k/n_i) |f|(x), \]
we have \( |\lambda R_\lambda f(x)| \leq \lim\inf f_i^*(x) \) a.e. for any rational \( \lambda > 0 \). Since the mapping \( \lambda \rightarrow \lambda R_\lambda f(x) \) is continuous for a.e. \( x \), it follows that
\[ \sup_{0<\lambda<\infty} |\lambda R_\lambda f(x)| = \sup_{\lambda > 0} |\lambda R_\lambda f(x)| \quad \text{a.e.} \]
\[ \lambda \text{ rational} \]
Thus
\[ f^*(x) \leq \lim \inf f_i^*(x) \quad \text{a.e.} \]

By Fatou's lemma and Lemma 5 we have
\[ \|f^*\| \leq \left( \frac{p}{p-1} \right) \|f\|. \]
This completes the proof.

6. **Theorem.** For any \( f \in L_p(\mu) \), the limit

\[
\lim_{\lambda \to 0^+} \lambda R_\lambda f(x)
\]

exists and is finite a.e.

**Proof.** The argument is the same as in [7]. For \( 1 < p < \infty \), \( L_p(\mu) \) is reflexive and thus the vector subspace of functions \( f \) of the form

\[
f = h + \sum_{i=1}^n [I - T(t_i)]g_i,
\]

where \( T(t)h = h \) for all \( t \geq 0 \) is dense in \( L_p(\mu) \) (see Corollary VIII. 7.2 in [3]). Since

\[
\lambda \int_0^\infty e^{-\lambda t} T(t) [I - T(t_i)] g_i(x) \, dt
\]

\[
= \lambda e^{\lambda t_i} \int_0^{t_i} e^{-\lambda t} T(t) g_i(x) \, dt + \lambda (1 - e^{\lambda t_i}) \int_0^\infty e^{-\lambda t} T(t) g_i(x) \, dt \quad \text{a.e.}
\]

for each \( i \), and

\[
\lim_{\lambda \to 0^+} \lambda e^{\lambda t_i} \int_0^{t_i} e^{-\lambda t} T(t) g_i(x) \, dt = 0 \quad \text{a.e.}
\]

for each \( i \), it follows from Lemma 5 that

\[
\lim_{\lambda \to 0^+} \lambda \int_0^\infty e^{-\lambda t} T(t) [I - T(t_i)] g_i(x) \, dt = 0 \quad \text{a.e.}
\]

for each \( i \). Thus \( \lim_{\lambda \to 0^+} \lambda R_\lambda f(x) \) exists and is finite for any \( f \) in a dense subset of \( L_p(\mu) \). Hence the Banach convergence theorem [3, Theorem IV.11.3] implies that \( \lim_{\lambda \to 0^+} \lambda R_\lambda f(x) \) exists and is finite a.e. for all \( f \in L_p(\mu) \). Q.E.D.

We remark that if we set \( \tilde{f} = \lim_{\lambda \to 0^+} \lambda R_\lambda f(x) \), then it follows from Lemma 5 and the Lebesgue dominated convergence theorem [3, III.6.16] that \( \tilde{f} \in L_p(\mu) \) and \( \lambda R_\lambda f(x) \) converges to \( \tilde{f} \) in norm as well as pointwise.

**Bibliography**


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