LARGE BASIS DIMENSION AND METRIZABILITY

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Abstract. In this paper it is proved that if $X$ is a regular Lindelöf space having finite large basis dimension, then $X$ is metrizable if and only if it is a $\Sigma$-space or a $\omega\Delta$-space.

Introduction. A collection $\Gamma$ of subsets of a set $X$ is said to have rank 1 if whenever $g_1, g_2 \in \Gamma$ and $g_1 \cap g_2 \neq \emptyset$, then $g_1 \subseteq g_2$ or $g_2 \subseteq g_1$. According to P. J. Nyikos [8], a topological space $X$ is said to have large basis dimension $\leq n$, denoted $\text{Bad } X \leq n$, if $X$ has a basis which is the union of $\leq n + 1$ rank 1 collections of open sets. (A. V. Arhangel'skii [1], [2], [3] uses the terminology “having a basis of big rank $\leq n + 1$” instead of “large basis dimension $\leq n$”.) $\text{Bad } X$ coincides with $\dim X$ and $\text{Ind } X$ for all metric spaces. For the case $n = 0$, the spaces become the nonarchimedean spaces of A. F. Monna [6].

It is the purpose of this paper to prove that every compact $T_2$-space having finite large basis dimension is metrizable. This answers a question of Arhangel'skii, first proposed in [2], where he proves that every compact nonarchimedean space is metrizable, and repeated in [3]. Some generalizations of the above result are also obtained.

Main result. All our spaces are assumed to be $T_1$. The main result of this paper is the following:

Theorem 1. Let $X$ be a regular Lindelöf space having finite large basis dimension. Then the following are equivalent:

(i) $X$ is metrizable,
(ii) $X$ is a $\Sigma$-space [7],
(iii) $X$ is a $\omega\Delta$-space [4].

As an immediate corollary, we have the result stated in the introduction:

Corollary 1. Every compact $T_2$-space having finite large basis dimension is metrizable.

By [3, Lemma 3], if $X$ has a basis $\Gamma = \bigcup \{ \Gamma_i \mid i = 1, 2, \ldots, n \}$ such that each $\Gamma_i$ is a rank 1 collection, then $X = \bigcup \{ X_i \mid i = 1, 2, \ldots, n \}$ where $X_i$ is such that $\Gamma_i$ contains a local basis of each point of $X_i$. Our method of proving Theorem 1 is to show that if $X$ is a regular Lindelöf space satisfying (ii) or (iii), then $X$ is first countable and each $X_i$ is Lindelöf. From this it will follow (see Lemma 3) that for each $X_i$ there is a point-countable collection of open
subsets of $X$ which contains a basis of each point of $X$; hence $X$ has a point-countable basis, and is therefore metrizable by known results. It will be helpful to establish some lemmas.

**Lemma 1.** Let $\Gamma$ be a rank 1 collection of open subsets of a space $X$ which contains a basis at a point $x_0 \in X$. Let $\Gamma' \subset \Gamma$ and suppose $x_0 \in \cap \Gamma'$. Then either $x_0 \in \text{Int}(\cap \Gamma')$ or $\{x_0\} = \cap \Gamma'$. In the latter case, either $\Gamma'$ contains a basis at $x_0$, or $x_0$ is an isolated point.

**Proof.** Suppose there exists $y \in \cap \Gamma'$, $y \neq x_0$. Choose $g \in \Gamma$ such that $x_0 \in g$ but $y \not\in g$. Then if $g' \in \Gamma'$, $g' \not\subset g$ and so $g \subset g'$. Hence $g \subset \cap \Gamma'$ and so $x_0 \not\in \text{Int}(\cap \Gamma')$.

Now suppose $\{x_0\} = \cap \Gamma'$, but that $\Gamma'$ does not contain a basis at $x_0$. Then there is some $g \in \Gamma$ which contains $x_0$ but does not contain any element of $\Gamma'$. Thus $g \subset g'$ for all $g' \in \Gamma'$, so $g \subset \cap \Gamma' = \{x_0\}$ and $x_0$ is therefore isolated.

Let $\Omega$ be the first uncountable ordinal, and let $P_\Omega$ be the space obtained from the ordinal space $[0,\Omega]$ by isolating all ordinals less than $\Omega$.

**Lemma 2.** Let $X$ be a regular Lindelöf space having finite large basis dimension. Then either $X$ is first countable, or $X$ contains a closed subspace homeomorphic to $P_\Omega$.

**Proof.** Suppose $X$ is not first countable at $x_0 \in X$. Let $\mathcal{U}_0$ be an open cover of $X - \{x_0\}$ such that for each $U \in \mathcal{U}_0$, $x_0 \not\in \overline{U}$. By [3, Theorem 1], $X$ is hereditarily metacompact. Let $\mathcal{V} = \{V_\alpha | \alpha \in A\}$ be a minimal point finite open refinement of $\mathcal{U}_0$. Since $\{x_0\} = \cap \{X - V_\alpha | \alpha \in A\}$, $\mathcal{V}$ must be uncountable, for otherwise it would follow from Lemma 1 that $x_0$ has a countable basis. Choose $x_\alpha \in V_\alpha - \bigcup \{V_\beta | \beta \neq \alpha\}$. Then $S' = \{x_\alpha | \alpha \in A\}$ has no cluster point in $X - \{x_0\}$. Since $X$ is Lindelöf, every neighborhood of $x_0$ contains all but countably many elements of $S'$.

Let $S = S' \cup \{x_0\}$. We claim that $S$ is homeomorphic to $P_\Omega$. To prove this, we need only show that if $C$ is an infinite subset of $S'$ such that $\text{card}(C) < \text{card}(S')$, then $C$ is closed in $S$. To this end, let $\Gamma$ be a rank 1 collection of open sets which contains a basis at $x_0$, and for each $c \in C$ choose $U_c \in \Gamma$ such that $x_0 \in U_c$ but $c \not\in U_c$. $U_C = \cap \{U_c | c \in C\}$ contains all but at most $\aleph_0 \cdot \text{card}(C) = \text{card}(C)$ elements of $S'$, and so by Lemma 1, $x_0 \in \text{Int}(U_C)$. Thus $C$ is closed and the proof is finished.

**Lemma 3.** Let $X$ be a first countable space, and let $X'$ be a subspace of $X$ such that some rank 1 collection $\Gamma_0$ of open subsets of $X$ contains a basis at each point of $X'$. Suppose also that $X'$ is Lindelöf. Then there exists a point-countable collection of open subsets of $X$ which contains a basis at each point of $X'$.

**Proof.** Let $\mathcal{C}$ be the set of all chains in $\Gamma_0$ (i.e., $C \in \mathcal{C}$ if $C$ is a subset of $\Gamma_0$ and is totally ordered by inclusion), and let $\Gamma = \{ \bigcup C | C \in \mathcal{C} \}$. By [9, Lemma 2], $\Gamma$ has rank 1. Let $\Gamma' = \{ g \cap X' | g \in \Gamma \}$. Clearly, $\Gamma'$ is the set of all unions of chains in $\Gamma_0 = \{ g \cap X' | g \in \Gamma_0 \}$. By [9, Lemma 2], the elements of $\Gamma'$ are clopen (open and closed) subsets of $X'$. By [9, Theorems 3 and 4], $X'$ can be partitioned into a collection $\mathcal{F}_\Omega$ of disjoint elements of $\Gamma'$. Furthermore we can ensure that this collection contains more than one element.
Clearly, any two elements of the corresponding collection \( \mathcal{U}_0 \) of elements of \( \Gamma \) are also disjoint.

We proceed to construct, for each \( \alpha < \Omega \), a collection \( \mathcal{U}_\alpha \) of disjoint elements of \( \Gamma \). Suppose \( \mathcal{U}_\alpha \) has been defined for all \( \alpha < \beta \). Let \( \mathcal{V}_\beta = \{ \cap_{\alpha < \beta} U_\alpha | U_\alpha \in \mathcal{U}_\alpha \} \), and let \( \mathcal{V}_\beta = \{ V \in \mathcal{V}_\beta | V \cap X' \text{ contains more than one point} \} \). By [3, Lemma 4], \( V \cap X' \) is clopen in \( X' \) whenever \( V \in \mathcal{V}_\beta \). Thus \( V \cap X' \) can be partitioned into a collection \( \mathcal{U}_V \) of (more than one) disjoint elements of \( \Gamma' \). Since \( V \cap X' \subset \text{Int}(V) \), we can ensure that every element of the corresponding collection \( \mathcal{U}_V \) of elements of \( \Gamma \) is contained in \( V \). Let \( \mathcal{U}_V = \bigcup \{ \mathcal{U}_V | V \in \mathcal{V}_\beta \} \).

Suppose \( V \in \mathcal{V}_\beta \), \( V \cap X' = \{ x_V \} \), and \( V \) contains more than one point of \( X \). Then \( x_V \in \text{Int}(V) \), and so there exists a local basis \( \{ g_n(V) \}_{n=1}^{\infty} \) of \( x_V \) such that \( g_n(V) \subset V \) for all \( n \). Let \( \mathcal{P}_\beta \) be the collection of all such \( g_n(V) \)'s.

Let \( \mathcal{W} = \bigcup \{ \mathcal{U}_\beta \cup \mathcal{P}_\beta | \beta < \Omega \} \). Since \( \mathcal{V}_\beta \) is a collection of disjoint sets, so is \( \mathcal{W} \); also, \( \mathcal{P}_\beta \) is point-countable, and \( \bigcup \mathcal{P}_\beta \cap \bigcup \mathcal{U}_\beta = \emptyset \).

We claim that \( \mathcal{W} \) is point-countable. Choose \( x_0 \in X \). If \( x_0 \in \bigcup \mathcal{P}_\beta \), then \( x_0 \notin \bigcup \{ \mathcal{U}_\alpha \cup \mathcal{P}_\alpha | \alpha > \beta \} \). In this case, then, \( x_0 \) belongs to at most countably many elements of \( \mathcal{W} \). Therefore, if \( x_0 \) is contained in uncountably many elements of \( \mathcal{W} \), then for each \( \alpha < \Omega \) there exists \( U_\alpha \in \mathcal{U}_\alpha \) such that \( x_0 \in U_\alpha \). By the way the \( U_\alpha \)'s were constructed, if \( \alpha < \alpha' < \Omega \), then \( U_\alpha \cap X' \subseteq U_{\alpha'} \cap X' \).

Let \( U_\alpha = \cap \{ U_\alpha | \alpha < \Omega \} \). \( U_\alpha \) cannot be clopen in \( X' \), for otherwise \( (X' \cap U_\alpha) \cup \{ X' - U_\alpha | \alpha < \Omega \} \) is an open cover of \( X' \) with no countable subcover.

However, if \( U_\alpha \cap X' \) is not clopen, then again by [3, Lemma 4], \( U_\alpha \cap X' = \{ x' \} \) for some \( x' \in X' \). For each \( \alpha < \Omega \), choose \( x_\alpha \in (U_\alpha \cap X') - (U_{\alpha+1} \cap X') \). It is easy to see that \( x' \) is the only cluster point of \( S = \{ x_\alpha | \alpha < \Omega \} \) in \( X' \). Since \( X' \) is Lindelöf, every neighborhood of \( x' \) must contain all but countably many elements of \( S \), contradicting the fact that \( X \) is first countable. Therefore \( \mathcal{W} \) is point-countable as claimed.

Choose \( x \in X' \). There exists a least ordinal \( \beta \) such that \( x \notin \bigcup \mathcal{U}_\beta \). Let \( \mathcal{U}_x = \{ U_\alpha | x \in U_\alpha \in \mathcal{U}_\alpha, \alpha < \beta \} \), and let \( \cap \mathcal{U}_x = V \in \mathcal{V}_\beta \). Then \( V \cap X' = \{ x \} \). Hence either \( x \in g_n(V) \), \( n = 1, 2, \ldots \) or \( V = \{ x \} \), whence \( x \) is discrete in \( X \) or \( \mathcal{U}_x \) contains a local basis at \( x \). Therefore \( \mathcal{W} \cup \{ x \in X' | x \text{ is discrete in } X \} \) is a point-countable collection of open subsets of \( X \) which contains a local basis at each point of \( X' \), and the proof is finished.

**Proof of Theorem 1.** The theorem is true if \( \text{Bad } X = 0 \) [8, Theorem 1.3]. Suppose it is true whenever \( \text{Bad } X < k - 1 \). Let \( X \) be a regular Lindelöf space with \( \text{Bad } X \leq k - 1 \), i.e., \( X \) has a basis \( \Gamma = \bigcup \{ \Gamma_i | i = 1, 2, \ldots, k + 1 \} \) where each \( \Gamma_i \) has rank 1.

Since a paracompact \( w\Delta \)-space is an \( M \)-space, and every \( M \)-space is a \( \Sigma \)-space, we need only prove that if \( X \) is a \( \Sigma \)-space, then \( X \) is metrizable. Since \( P_1 \) is not a \( \Sigma \)-space, by Lemma 2 \( X \) is first countable. Let \( X_i \) be the subspace of \( X \) such that \( x \in X_i \) if and only if \( \Gamma_i \) contains a basis at \( x \). We need only prove that \( X_i \) is Lindelöf, for then we can apply Lemma 3 to each \( X_i \) to show that \( X \) has a point-countable basis, from which it follows that \( X \) is metrizable [10].

\( X \) is a nonarchimedean space, hence paracompact [9, Theorem 4]. Therefore if \( X_1 \) is not Lindelöf, there is an uncountable subset \( T \) of \( X_1 \) which has no
cluster point in $X_1$. Consider the closure $\overline{T}$ of $T$ in $X$. The points of $T$ are discrete in $\overline{T}$, so $\bigcup \{T_2 \cup T_3, \ldots, T_{k+1}\}$ contains a basis (in the subspace $\overline{T}$) for each point of $\overline{T}$. Thus $\text{Bad} \overline{T} k - 1$, so by the induction hypothesis, $\overline{T}$ is metrizable. Thus $\overline{T} - T$ is $G_2$ in $\overline{T}$, and so there exists an uncountable subset $T'$ of $T$ which is closed in $\overline{T}$, and therefore in $X$, contradicting the fact that $X$ is Lindelöf. Thus $X_1$ is Lindelöf. Similarly, $X_2, \ldots, X_{k+1}$ are Lindelöf, and the proof is finished.

REFERENCES


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