AN ALMOST CONTINUOUS FUNCTION
\( f : S^n \to S^m \) WHICH COMMUTES WITH THE ANTIPODAL MAP

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Abstract. It is shown that if \( n, m \geq 1 \) are integers, then there exists an almost continuous function from the \( n \)-sphere \( S^n \) onto \( S^m \) which commutes with the antipodal map.

Introduction. Hunt [1] has generalized the Borsuk-Ulam antipodal point theorem by proving that no connectivity function \( f : S^n \to S^{n-1} \) commutes with the antipodal map. Since, if \( n > 1 \), by Corollary 1 of Stallings [5], such a function is almost continuous, it seems reasonable to ask whether Hunt's result holds for almost continuous functions. The purpose of this note is to give a counterexample.

Definitions and conventions. In the sequel we regard a function as being identical with its graph.

Suppose \( f : X \to Y \). That \( f \) is almost continuous means that if \( f \subset D \), where \( D \) is an open subset of \( X \times Y \), then there exists a continuous function \( g : X \to Y \) such that \( g \subset D \). That \( K \) is a minimal blocking set of a non-almost continuous function \( f \) means that \( K \) is a closed subset of \( X \times Y \), \( K \cap f = \emptyset \), \( K \cap g \neq \emptyset \) whenever \( g : X \to Y \) is continuous, and no proper subset of \( K \) has the preceding properties.

We denote by \( S^n \) the set of all points \( x = (x_1, x_2, \ldots, x_{n+1}) \) of Euclidian \( (n + 1) \)-space \( \mathbb{R}^{n+1} \) such that \( (\sum_{i=1}^{n+1} x_i^2)^{1/2} = 1 \). A function \( f : S^n \to S^m \) is said to commute with the antipodal map if \( f(-x) = -f(x) \) for each \( x \) in \( S^n \).

The natural projection map of \( X \times Y \) onto \( X \) is denoted by \( p : X \times Y \to X \). The letter \( c \) denotes the cardinality of the real line.

The main results.

Theorem 1. Suppose \( f : X \to S^m \) is not almost continuous where \( m \geq 1 \) and \( X \) is a compact metric space. There exists a minimal blocking set \( K \) of \( f \) and \( p(K) \) is a perfect set.

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Proof. That $K$ exists follows from Theorem 2 of [3]. Assume that $z$ is an isolated point of $p(K)$ and let $U$ be a neighborhood of $z$ such that $U \cap p(K) = \{z\}$. Note that $p(K) \neq \{z\}$, because otherwise the constant map $g: X \to S^m$ such that $g(x) = f(z)$ would not intersect $K$. Thus $K - (p^{-1}(z) \cap K)$ is a closed, proper subset of $K$. By the minimality of $K$ there exists a continuous function $g: X \to S^m$ such that $p(K \cap g) = \{z\}$. Let $y$ be a point of $S^m$ different from $f(z)$ and $g(z)$ and let $V$ be a neighborhood of $z$ such that $V \subset U$ and $g(V) \subset S^m - \{y\}$. Since $S^m - \{y\}$ is homeomorphic to $R^m$, it is an AR [4, p. 339], so the continuous function $h: (V - V) \cup \{z\} \to S^m - \{y\}$, defined by $h(V - V) = g(V - V)$ and $h(z) = f(z)$, has a continuous extension $h': V \to S^m - \{y\}$. But then $g' = g(X - V) \cup h'$ is a continuous function from $X$ into $S^m$ and $g' \cap K = \emptyset$, a contradiction. Thus $p(K)$ has no isolated points and is a perfect set.

Theorem 2. Suppose $n$ and $m$ are integers with $n, m \geq 1$. There exists an almost continuous function $f: S^n \to S^m$ which commutes with the antipodal map.

Proof. Denote by $\theta$ the set of all closed subsets $T$ of $S^n \times S^m$ such that $\text{card}(p(T)) = c$. It follows from Theorem 1 that if $f: S^n \to S^m$ intersects each member of $\theta$, then $f$ is almost continuous. Using transfinite induction in a manner quite similar to the proof of Theorem 2 of [2], for each $T$ in $\theta$ we may choose $x_T$ in $p(T)$ and define $f(x_T)$ and $f(-x_T)$ so that $(x_T, f(x_T))$ is in $T$ and $f(-x_T) = -f(x_T)$. Now, if $x$ is a point of $S^n$ such that $f(x)$ is not defined by the above induction, neither is $f(-x)$ defined. So, for each such $x$, we may define $f(x)$ and $f(-x)$ arbitrarily so long as $f(-x) = -f(x)$. This completes the proof.

References

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