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APPLICATIONS OF INTERSECTION THEORY

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ABSTRACT. A Borsuk-Ulam type theorem for semifree actions on S^{2n} is obtained via a formula of Lefschetz.

One version of the classical Borsuk-Ulam Theorem states that if T is a fixed point free involution of S^n and f: $S^n \to S^n$ is of even degree, then there is an x such that $f(x) = f(T(x))$. [Conner and Floyd: Theorem 33.1]. In this note, using an argument of Lefschetz, we prove a similar theorem for semifree actions, i.e. differentiable group actions which are free outside the fixed point set.

THEOREM. *Let G be a compact Lie group acting semifreely and preserving orientation on S^{2n} with fixed point set $\{x, y\}$. Let $f = S^{2n} \to S^{2n}$ be differentiable such that f is a local diffeomorphism near x and y and $|\deg f| \geqslant 2$; then if $1 \neq g \in G$ there exists a nonfixed point z such that $f(z) = f(g(z))$.*

Before we prove this Theorem, we note that the condition on being a local diffeomorphism near x and y is necessary. Otherwise if $(p, q) = 1$, and $G = Z_p$ is the action on S^2 gotten by suspending differentiably the usual Z_p action on S^1 and f: $S^2 \to S^2$ is the map of degree q gotten by suspending the q-fold covering of S^1, then the conclusion of the theorem fails. We also note that the condition on g being a diffeomorphism induced by a group action is more restrictive than we need, in that we only need that 1 is not an eigenvalue of dg at x or y.

PROOF OF THEOREM. Consider the intersection of ΔS^{2n} and

$$(f \times f \circ g)(\Delta S^{2n}).$$

If the Theorem were false, the geometric intersection would consist of only the points $(f(x), f(x))$ and $(f(y), f(y))$. At each of these points the local algebraic intersection number is ± 1, since f was assumed to be a local diffeomorphism at x and y, and the local fixed point index of g at x and y is ± 1; thus the total algebraic intersection number of these 2 cycles is 0 or ± 2. According to [Lefschetz], this number is equal $L(f \circ g, f)$, which in this case is $2 \deg f$.

One application of this Theorem is the nonexistence of smooth embeddings of S^2 in $S^2 \times S^2$ of the form (f, k), where $f(z) = z^m$ and degree k is n such

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that $|m|$ and $|n|$ are $\geqslant 2$. Here G is the ordinary Z_n action on S^2, multiplication by the nth roots of unity.

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