APPLICATIONS OF INTERSECTION THEORY

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Abstract. A Borsuk-Ulam type theorem for semifree actions on $S^{2n}$ is obtained via a formula of Lefschetz.

One version of the classical Borsuk-Ulam Theorem states that if $T$ is a fixed point free involution of $S^n$ and $f: S^n \to S^n$ is of even degree, then there is an $x$ such that $f(x) = f(T(x))$. [Conner and Floyd: Theorem 33.1]. In this note, using an argument of Lefschetz, we prove a similar theorem for semifree actions, i.e. differentiable group actions which are free outside the fixed point set.

Theorem. Let $G$ be a compact Lie group acting semifreely and preserving orientation on $S^{2n}$ with fixed point set $\{x, y\}$. Let $f = S^{2n} \to S^{2n}$ be differentiable such that $f$ is a local diffeomorphism near $x$ and $y$ and $|\text{deg } f| > 2$; then if $1 \neq g \in G$ there exists a nonfixed point $z$ such that $f(z) = f(g(z))$.

Before we prove this Theorem, we note that the condition on being a local diffeomorphism near $x$ and $y$ is necessary. Otherwise if $(p, q) = 1$, and $G = \mathbb{Z}_p$ is the action on $S^2$ gotten by suspending differentiably the usual $\mathbb{Z}_p$ action on $S^1$ and $f: S^2 \to S^2$ is the map of degree $q$ gotten by suspending the $q$-fold covering of $S^1$, then the conclusion of the theorem fails. We also note that the condition on $g$ being a diffeomorphism induced by a group action is more restrictive than we need, in that we only need that $1$ is not an eigenvalue of $dg$ at $x$ or $y$.

Proof of Theorem. Consider the intersection of $\Delta S^{2n}$ and $(f \times f \circ g)(\Delta S^{2n})$.

If the Theorem were false, the geometric intersection would consist of only the points $(f(x), f(x))$ and $(f(y), f(y))$. At each of these points the local algebraic intersection number is $\pm 1$, since $f$ was assumed to be a local diffeomorphism at $x$ and $y$, and the local fixed point index of $g$ at $x$ and $y$ is $\pm 1$; thus the total algebraic intersection number of these 2 cycles is 0 or $\pm 2$. According to [Lefschetz], this number is equal $L(f \circ g, f)$, which in this case is $2 \text{deg } f$.

One application of this Theorem is the nonexistence of smooth embeddings of $S^2$ in $S^2 \times S^2$ of the form $(f, k)$, where $f(z) = z^m$ and degree $k$ is $n$ such

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that $|m|$ and $|n|$ are $\geq 2$. Here $G$ is the ordinary $Z_n$ action on $S^2$, multiplication by the $n$th roots of unity.

REFERENCES
