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A PECULIAR TRIANGULATION OF THE 3-SPHERE[1]

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ABSTRACT. A triangulation of the 3-sphere with 10 vertices is presented, which is not directly obtainable by a generalized stellar subdivision from any 3-sphere with 9 vertices. This answers in the affirmative a conjecture by B. Grünbaum. This example is shown to be the minimal possible.

1. **Grünbaum's conjecture.** The term *combinatorial n-sphere* is used here in its sense in [1] and [7], i.e., it is an abstract simplicial n-complex which, when geometrically realized (e.g. in the euclidean space R^{2n+1}, see [6, p. 5]), is a triangulation of the topological n-sphere S^n. Thus, each combinatorial n-sphere S has a certain number $f_i(S)$ of i-simplices for each i, $0 \leqslant i \leqslant n$. Clearly $f_0(S) \geqslant n+2$. It is the purpose of the present work to present a combinatorial 3-sphere with 10 vertices which is not *directly obtainable* (defined below), thus solving in the affirmative a conjecture by Grünbaum [7, Conjecture 5].

Since all the complexes and spheres mentioned in the present work are combinatorial, we shall omit the adjective combinatorial. We use the terminology of [3] and [4], and bear in mind that an n-sphere is a particular case of a (combinatorial) n-manifold as defined there. The n-simplex whose vertices are $a_0, a_1, \ldots, a_n$ is denoted $a_0 a_1 \cdots a_n$.

A simple and natural way to obtain a 3-sphere S_{n+1} with $n+1$ vertices from a 3-sphere S_n with n vertices, is to apply on S_n a generalized stellar subdivision, as defined in [7]. In this case we say that S_{n+1} is directly obtainable from S_n. More precisely:

Following Alexander [1], we define an m-element to be any simplicial m-complex C whose body is homeomorphic to a topological n-simplex Δ_m. If ϕ is that homeomorphism, then the *boundary* bd C of C is the complex of all the simplices $\Delta \in C$ such that $\phi(\Delta)$ lies on the boundary of Δ_m. It is clear that bd C is independent of the particular homeomorphism ϕ.

Let S be an m-sphere with $n+1$ vertices ($m \geqslant 2, n \geqslant m+2$). We would like to remove from S one vertex $v \in S$ together with the complex $\operatorname{star}(v, S)$, and then complete the complex $S - \operatorname{star}(v, S)$ to an m-sphere S' with n vertices. The "hole" created in S by removing $\operatorname{star}(v, S)$ should be replaced by

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some m-element C such that bd C = link(v, S), all the vertices of C are in bd C, and S' = antistar $(v, S) \cup C$ is an m-sphere.

If this can be done, we say that S can be *refilled* at the vertex v, and that C is the *refill*. In this case, we also say that S is *directly obtainable at the vertex v* from S', since S can be obtained from S' by replacing the subcomplex C of S' by a new complex which has one new vertex v, and which will be cl star(v, S). The last operation that yields S from S' is the generalized stellar subdivision of S' mentioned in [7].

Notice that, as follows easily from [1, Theorem 12.5], if the m-sphere S can be refilled at a vertex $v \in S$ and if C is the refill, then $C \cap$ antistar (v, S) = link(v, S), i.e., no interior simplex of C is in antistar (v, S).

If S and S' are m-spheres with $n + 1$ and n vertices, respectively, and there is a vertex in S at which S is directly obtainable from (an m-sphere combinatorially isomorphic to) S', we say that S is *directly obtainable from* S'. If S is an m-sphere and there is some m-sphere S' from which S is directly obtainable, we say that S is *directly obtainable*. It is easy to see that every 2-sphere with $n > 4$ vertices is directly obtainable.

The above mentioned conjecture by Grünbaum can now be stated as follows:

GRÜNBAUM'S CONJECTURE . *For suitable* $m \geqslant 3$ *and* $n \geqslant m + 3$ *there exists an m-sphere with n vertices which is not directly obtainable.*

2. **A nondirectly obtainable 3-sphere**. Let S be the simplicial 3-complex with 10 vertices 0, 1, . . . , 9 composed of the following 35 3-simplices and their faces as shown in the list. We claim that S is a 3-sphere which is not directly obtainable.

0168	0246	1234	2456	1689
0178	0359	1357	2567	1289
0157	0349	1347	2579	3689
0135	0459	1478	2679	3489
0123	0456	1248	3679	2589
0126	0567	3478	3678	2458
0234	0678	3579	1269	4589

The first thing to notice is the fact that S is *neighborly*, i.e., for every two vertices $x, y \in S$, the edge (1-simplex) xy is also in S. Next notice that for every vertex $x \in S$, link(x, S) is a 2-sphere, and therefore S is a 3-manifold in the sense of [3]. Moreover, for every vertex $x \in S$, the 2-sphere link(x, S) is combinatorially isomorphic to the 2-sphere shown in the Figure (while link$(0, S)$ is precisely the 2-sphere in the Figure). The 2-sphere shown in the Figure is not stacked (see [3, Definition 2.1]; a stacked 2-sphere is essentially a dissection of a 3-ball as defined in [5]). Now, Theorem 2.4 of [3] formulated to suit our case states that if M is a neighborly 3-sphere directly obtainable at a vertex $x \in M$, then link(x, M) is a stacked 2-sphere. Hence, assuming for the moment that S is a 3-sphere, we conclude that S is not directly obtainable.

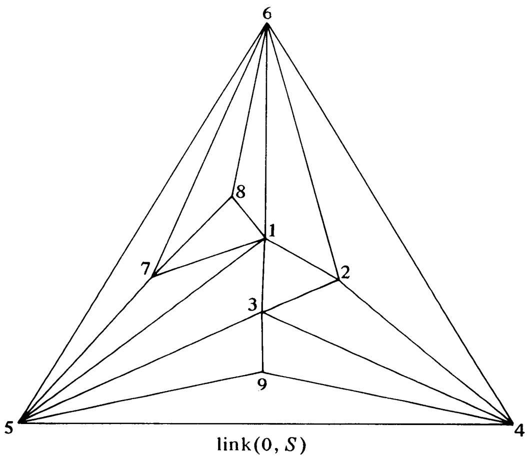
How can one prove that the 3-manifold S is indeed a 3-sphere? It is easy to check that S is connected and orientable. Since $\operatorname{star}(0, S)$ is a 1-connected subcomplex of S containing all the vertices of S, we can use Theorem 6.3.6 of [8] to compute the fundamental group (i.e., the first homotopy group) of S, and find that S is simply connected. However, in the lack of a proof for the famous Poincaré Conjecture, it does not prove that S is a 3-sphere.

The following construction yields that S is indeed a 3-sphere. Start with the 3-element $\operatorname{star}(0, S)$, i.e., the complex composed of the first 14 3-simplices in the list, and their faces. Its boundary is $\operatorname{link}(0, S)$ and is shown in the Figure. Add to it the remaining 21 3-simplices of S, one at a time, *according to their order in the list*. In each step, except for the very last, we obtain a 3-element. The first step, e.g., which adds the 3-simplex 1234 to $\operatorname{star}(0, S)$, yields a 3-element whose boundary is obtained from the Figure by removing the edge 23 and adding the edge 14. The step before the last, adding the 3-simplex 2458, yields a 3-element C whose boundary is the boundary complex of the 3-simplex 4589. The last step, which adds the 3-simplex 4589, closes C to a 3-sphere (see [1, Theorem 14.1], also [10, Proposition 1.2]).

This construction yields also that the sphere S is *shellable*, i.e., its 35 3-simplices can be ordered $c_1, c_2, \ldots, c_{35}$ so that for each integer $k \leqslant 34$, $c_1 \cup c_2 \cup \cdots \cup c_k$ is a 3-element. The question whether or not there exists a nonshellable sphere—not to be confused with the question of the existence of a nonshellable cell, which is already solved in the affirmative—is still open, in spite of the contention of McMullen and Shephard [9, p. 177]. (A triangulated 3-sphere D is unshellable iff for *every* 3-simplex T in D, $D - T$ is an unshellable cell, see also [10].)

Since all the 3-spheres with n vertices $(5 < n < 10)$ are known to be directly obtainable (see [2], [3] and [4]), our 3-sphere S is the "smallest" 3-sphere that is not directly obtainable.

The fact that all ten links $\operatorname{link}(x, S)$ (x is a vertex in S) are of the same type already indicates that S possesses a high degree of symmetry. Indeed, the permutation $\psi = (0, 2, 8, 3, 5, 6, 1, 4, 9, 7)$ of the ten vertices of S induces a combinatorial automorphism of S, and using powers of ψ it is clear that for every two vertices $x, y \in S$, there is a combinatorial automorphism of S that maps x to y.



$\operatorname{link}(0, S)$

FIGURE

For the sake of precision, we formalize our result as a theorem.

THEOREM. *There exists a (triangulation of the) 3-sphere which is not directly obtainable. The minimum number of vertices in such a 3-sphere is* 10. *There exists a 3-sphere with* 10 *vertices which is neighborly, shellable, and not directly obtainable.*

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