AXIAL MAPS WITH FURTHER STRUCTURE

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Abstract. For $F = R, C$ or $H$ an $F$-axial map is defined to be an axial map $\mathbb{R}P^n \times \mathbb{R}P^n \to \mathbb{R}P^{n+k}$ equivariant with respect to diagonal and trivial $F^*$-actions. Analogously to the real case, it is shown that $C$-axial maps correspond to immersions of $CP^n$ in $\mathbb{R}^{2n+k}$ while (for $F = R$ and for $F = C$, $k$ odd) embeddings induce $F$-symmaxial maps. Examples are thereby given of symmaxial maps not induced by embeddings of $RP^n$, and of $R$-axial maps which are not $C$-axial. Furthermore, the relationships which hold when $F = R, C$ are no longer valid for $F = H$.

Let $F$ be one of the fields $R, C$ or $H$ of dimension $d (=1,2,4$ respectively) over $R$, whose units $F^*$ act on the right on $S(F^{n+1})$ to induce the projective space $FP^n$. Since the action of $R^*$ extends to the action of $F^*$, we may regard $F^*$ as acting also on $RP^n$ and thence diagonally on $RP^n \times RP^n$, $n \equiv -1 \ (d)$. By way of generalisation of the usual definitions ($F = R$—see [2], [4], [12]), we say $f: RP^n \times RP^n \to RP^{n+k}$ is $F$-axial of type $(N, k)$ if $f$ restricts to homotopy essential maps on the axes of the product and is equivariant with respect to the above $F^*$-action on its domain and trivial $F^*$-action on its range. If further $f$ is homotopy equivariant—through an $F^*$-equivariant homotopy—with respect to interchanging the factors of the domain and trivial $Z_2$-action on the range, $f$ is $F$-symmaxial. (When $F = R$ it is sometimes omitted from the notation.) This note explores the relationship between $F$-axial (resp. $F$-symmaxial) maps and the existence of an immersion (resp. embedding) of $FP^n$ in $R^n$, denoted $FP^n \subseteq (m)$ (resp. $FP^n \subset (m)$).

1. Theorem. Let $F = R$ or $C$, with $N = n$ or $(2n + 1)$ respectively.
(a) If $FP^n \subseteq (dn + k)$, then there exists an $F$-axial map of type $(N, k)$.
(b) If $FP^n \subset (dn + k)$, then there exists an $F$-symmaxial map of type $(N, k)$, provided $k$ is odd if $F = C$.
(c) If $FP^n \subset (dn + k)$, then the $F$-axial maps given by the constructions of (a) and (b) are homotopic through an $F^*$-equivariant homotopy.
(d) If there exists an $F$-axial map of type $(N, k)$ with $2k \geq dn + 1$, then $FP^n \subseteq (dn + k)$.

Proof. (a),(d). Let $\gamma$ be the realisation of the Hopf line bundle, $e$ the trivial real line bundle, and $\tau$ the real tangent bundle over $FP^n$. In the following sequence of implications, † indicates the use of the condition $2k \geq dn + 1$.

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\[ FP^n \subseteq (dn + k) \iff \tau \text{ is a subbundle of } (dn + k)e \quad [6] \]
\[ \iff \tau \oplus de = (n + 1)\gamma^* \text{ is a subbundle of } (dn + k + d)e \quad [7, \text{ p. 100}] \]
\[ \iff \text{there exists a skew map } (n + 1)\gamma^* \to (d(n + 1) + k)e \quad [5, (1.2)] \]
\[ \iff \text{there exists a map } S^N \times S^N \to S^{N+k} \text{ which induces an } F\text{-axial map of type } (N, k). \]

(b) Let \( f: FP \to \mathbb{R}^{dn+k} \) be an embedding. (To use conventional matrix notation, we shall assume here that \( F^* \) acts on \( \mathbb{R}^{dn} \) on the left.) Write \( \mathbb{R}_0^{n} = \mathbb{R}^{m} \setminus \{0\}; \quad \rho: \mathbb{R}_0^{n} \to S^{m-1}, \quad x \mapsto x/\|x\|; \quad \pi: S^{N} \to FP^n, \) and set \( \Delta = \{(x,wx) \in \mathbb{R}_0^{n+1}; \; w \in F^*\}, \) \( \Delta' = \Delta \cap (S^N \times S^N), \) \( e = (1,0, \ldots, 0) \in \mathbb{R}^{N+1+k}, \) and \( j: \mathbb{R}^{dn+k} \to \mathbb{R}^d \oplus \mathbb{R}^{dn+k} \) for the inclusion of the orthogonal complement of \( F_e \) in \( \mathbb{R}^{n+1+k}. \) For \( u,v \in S^N, \) write \( a = \langle \rho(u), \rho(v) \rangle; \) and define

\[ G: (S^N \times S^N, S^N \times S^N \setminus \Delta') \times I \to (\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}, \mathbb{R}_0^{n+1} \times \mathbb{R}_0^{n+1} \setminus \Delta), \]
\[ G(u,v,t) = \begin{bmatrix} 1 - |a|^2t^2 & -at \\ at & 1 \end{bmatrix} \begin{bmatrix} u \\ v - au \end{bmatrix}; \]
\[ g: (\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}, \mathbb{R}_0^{n+1} \times \mathbb{R}_0^{n+1} \setminus \Delta) \to (\mathbb{R}^{dn+k}, \mathbb{R}_0^{dn+k}). \]

Hence, define

\[ F: S^N \times S^N \times I \to S^{N+k}, \quad F(u,v,t) = \rho((ae + jgG(u,v,t)). \]
The reader may verify that these maps behave as required, so that \( F_0: S^N \times S^N \to S^{N+k} \) induces an \( F\text{-symmaxial map of type } (N,k). \) (When \( F = \mathbb{C}, \) the involution on \( \mathbb{C}F^{2n+1+k} \) given by \( \pm(ae + jz) \mapsto \pm(a\bar{e} + j\bar{z}) \) is homotopic to the identity provided \( k \) is odd.)

(c) Clearly it suffices to establish that the tangent bundle monomorphism

\[ \tau(f): \tau FP^n \to \tau \mathbb{R}^{dn+k} = \mathbb{R}^{dn+k} \times \mathbb{R}^{dn+k} \]
is fibre-homotopic to

\[ g': \tau FP^n \to \mathbb{R}^{dn+k} \times \mathbb{R}^{dn+k}, \quad g'(\begin{bmatrix} x \cr y \end{bmatrix}. F^*) = (f_{\pi}(x), g(x,y)) \]
\( (f, g \text{ as in (b))}, \) since the \( F\text{-axial maps of both (a) and (b) come from composition with } G_0: S^N \times S^N \to \pi* \tau FP^n \text{ specified in (b).} \)

But this is evident from the following homotopy (cf. \( [5, \text{ Lemma 2.2}]\)):

\[ H: \tau FP^n \times I \to \mathbb{R}^{dn+k} \times \mathbb{R}^{dn+k}, \]
\[ H((\begin{bmatrix} x \cr y \end{bmatrix}. F^*), t) = \left( f_{\pi}(x), \frac{f_{\pi}(x) + f_{\pi}(y)}{2(1 - t^2)} \right). \]
(\( \text{Note that, as } t \to 1, \quad 1 - t^2 = 2(1 - t) + O((1 - t)^2). \))

By \( [2], \) the numerical condition of 1(d) is satisfied when \( n > 7 \) if \( F = \mathbb{C} \) and may be omitted if \( F = \mathbb{R}. \) Thus 1(a),(d) yield that \( CP^n \subseteq (2n + k) \) implies
\[ P^{2n+1} \subseteq (2n + k + 1) \text{--cf. [12, (5.2)]}. \] When \( F = \mathbb{R} \), \( 1(b), (c) \) answer affirmatively a question raised in [2] (for which, I understand, Professors Feder and Gitler also have a proof); we now show the converse is not true.

2. Example. Let \( n \) be a power of 2. Then by [8], \( CP^n \subset (4n - 1) \); \( 1(b) \) now implies the existence of a C-symmaxial (and so \( R \)-symmaxial) map of type \((2n + 1, 2n - 1)\). But [9], [10] \( RP^{2n+1} \not\subset (4n) \), so that the existence of a symmaxial map of type \((n, k)\) does not imply \( RP^n \subset (n + k) \).

The next result is perhaps more predictable. Nevertheless, it illustrates the falsity of the converse to [12, (5.2)].

3. Example. Let \( n + 1 = 2^r \), where \( r \equiv 2, 3 \text{ (4)} \). Then by [4] \( RP^{2n+1} \subset (4n - 2r) \); so by [11] there exists an \( R \)-axial map of type \((2n + 1, 2n - 2r - 1)\). However, by [13], \( CP^n \not\subset (4n - 2r - 1) \), whence, from \( 1(c) \), the existence of an \( R \)-axial map of type \((2n + 1, k)\) does not imply the existence of a C-axial map of type \((2n + 1, k)\).

Since 1 shows that the situation for \( RP^n \) largely carries over to \( CP^n \), one might naively hope that a comparable result holds for \( HP^n \). However, [3, §4] casts doubt upon, and 5 below puts paid to, such hopes.

4. Lemma. If there exists an \( H \)-axial (resp. \( H \)-symmaxial) map \( f \) of type \((4n + 3, k)\), then there exists a C-axial (resp. C-symmaxial) map \( g \) of type \((4n + 3, k)\).

Proof. Write \( R^{4n+4} = C^{2n+2} \oplus C^{2n+2} \) which we identify with \( H^{n+1} \) as \( C^{2n+2} \oplus C^{2n+2} \). For \( x_i, y_i \in C^{2n+2}, i = 1, 2 \), \( f \) induces \( g \) by setting \( g(\pm (x_1, x_2), \pm (y_1, y_2)) = f(\pm (x_1 + \overline{x}_2, \pm (y_1 + \overline{y}_2)) \), since \((x_1a + (\overline{x}_2a)) = (x_1 + \overline{x}_2)a \text{ for } a \in C^* \). If \( f \) is symmaxial then clearly \( g \) is too.

5. Example. Let \( n \) be a power of 2. From [8], \( HP^n \subset (8n - 3) \). But if there were an \( H \)-symmaxial--or even \( H \)-axial--map of type \((4n + 3, 4n - 3)\), then by 4 above there would exist a C-axial map of type \((4n + 3, 4n - 3)\). So by \( 1(c) \) \( CP^{2n+1} \subset (8n - 1) \), which is contradicted by [1], [13]. Hence, \( HP^n \subset (4n + k) \) does not imply the existence of an \( H \)-axial map of type \((4n + 3, k)\).

As for positive results in the quaternionic case, we must content ourselves with the following observation.

6. Note. If there exists an \( H \)-axial map of type \((4n + 3, k)\) with \( 2k > 4n + 1 \), then \( HP^n \subset (4n + 3 + k) \). The proof is as for \( 1(c) \) above, save that one uses the characterisation of the tangent bundle given in [3, §4].

REFERENCES


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