THE ASYMPTOTIC EXPANSION OF THE ZETA-FUNCTION OF A COMPACT SEMISIMPLE LIE GROUP

ROBERT S. CAHN

Abstract. If $G$ is a connected, simply connected, semisimple Lie group with metric given by the negative of the Killing form and zeta-function $Z(t)$, then

$$Z(t) = \frac{\text{Vol } G}{(4\pi t)^{\dim G/2}} \exp|\delta|^2 t + \text{exponentially small error as } t \downarrow 0.$$ 

0. Introduction. Let $G$ be a compact, connected, simply connected, semisimple Lie group with Lie algebra $\mathfrak{g}$. As in [2] the metric $g$ on $G$ will be obtained by left translation of the negative of the Killing form of $\mathfrak{g}$ to the entire group. The Laplacian of $G$ is then the negative of the Casimir operator and the zeta-function of $(G, g)$ is

$$(0) \quad Z(t) = \frac{1}{|w|} \sum_{\Lambda \in L} f^2(\Lambda) \exp[-(|\Lambda|^2 - |\delta|^2)t].$$

In (0), $f(\Lambda) = \prod_{\alpha > 0}(\Lambda, \alpha)/\prod_{\alpha > 0}(\delta, \alpha)$, $\delta = \frac{1}{2} \sum_{\alpha > 0} \alpha$, $L$ is the lattice of integral weights, $|w|$ is the order of the Weyl group and inner products and norms are with respect to the Killing form. We will prove the following

Theorem.

$$Z(t) = \frac{\text{Vol } G}{(4\pi t)^{\dim G/2}} \exp|\delta|^2 t + \text{exponentially small error as } t \downarrow 0. $$

1. Theta relations. In this section we will state some well-known theta relations with minor modifications that are necessary for our purposes. Let $\mathbb{R}^n$ be Euclidean $n$-space endowed with the usual inner product. Then the Laplacian on $\mathbb{R}^n$, $\Delta$, is just $\partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$ where $(x_1, \ldots, x_n)$ is an orthonormal basis. We assume $P_q(x)$ is a homogeneous polynomial of degree $q$ such that $\Delta P_q \equiv 0$.

Lemma 1.

$$\sum_{m \in \mathbb{Z}^n} P_q(m) \exp(-t\pi|m|^2) = i^q t^{n/2-q} \sum_{m \in \mathbb{Z}^n} P_q(m) \exp(-\pi|m|^2/t).$$

Presented to the Society, January 23, 1975; received by the editors November 12, 1974.

AMS (MOS) subject classifications (1970). Primary 22C05, 43A75.

Key words and phrases. Compact semisimple Lie group, zeta-function, theta function.
Proof. This result is found in [1]. If \( p_q \equiv 1 \) we obtain the classical theta relation. It is important to remark that if \( q \geq 1 \) then \( p_q(0) = 0 \) and the right-hand side of (1) is ES (exponentially small) as \( t \downarrow 0 \).

Lemma 1 may be modified by taking a general lattice. If \( L \) is a lattice and \( L' \) is its dual then

Lemma 2.

\[
(2) \quad \sum_{m \in L} p_q(m) \exp(-\pi t |m|^2) = V_{L-1}^{-1} i^{\frac{q}{2}} t^{-\frac{n}{2-q}} \sum_{m \in L'} p_q(m) \exp(-\pi |m|^2/t)
\]

where \( V_L \) = volume of a fundamental parallelepiped of \( L \).

Proof. Lemma 2 follows directly from the Poisson summation formula.

We now proceed to apply Lemma 2 to the problem at hand.

2. Preliminary steps. We now let \( G \) be a fixed group and \( n \) be the rank of \( G \). We fix a Cartan subalgebra of \( g_C = g \otimes \mathbb{C} \), \( \tau \), and identify \( \mathbb{R}^n \) with the real span of the roots of \( g_C \) in \( \tau \), the complex dual of \( \tau \). \( L \) will be the lattice of integral weights. The natural inner product on \( \mathbb{R}^n \) will be given by the Killing form. With respect to this inner product, we pick an orthonormal basis \( \{x_1, \ldots, x_n\} \) and, as before, \( \Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2 \). We now consider 2 cases: \( n = 1 \) and \( n > 1 \).

3. Rank \( n = 1 \). If Rank \( G = 1 \), \( g \) is \( A_1 \) and \( f(\Lambda) \) is particularly simple. We may take \( f(\Lambda) = \Lambda \) and \( L = \mathbb{Z} \). Then

\[
Z(t) = \frac{1}{2} \exp(\delta^2 t) \sum_{\Lambda \in \mathbb{Z}} \Lambda^2 \exp(-|\Lambda|^2 t) \quad \text{with } |\Lambda|^2 = 2\Lambda^2.
\]

But

\[
\sum_{\Lambda \in \mathbb{Z}} \Lambda^2 \exp(-2\Lambda^2 t) = -\frac{d}{dt}\left( \frac{1}{2} \sum_{\Lambda \in \mathbb{Z}} \exp(-2\Lambda^2 t) \right)
\]

\[
= -\frac{d}{dt}\left( \frac{\pi^{1/2}}{2^{3/2} t^{1/2}} \sum_{\Lambda \in \mathbb{Z}} \exp(-\Lambda^2 \pi^2/2t) \right)
\]

\[
= \frac{\pi^{1/2}}{2^{5/2} t^{3/2}} \sum_{\Lambda \in \mathbb{Z}} \exp(-\pi^2 \Lambda^2/2t) - \frac{\pi^{5/2}}{2^{5/2} t^{5/2}} \sum_{\Lambda \in \mathbb{Z}} \Lambda^2 \exp(-\Lambda^2 \pi^2/2t)
\]

\[
= \frac{\pi^{1/2}}{2^{5/2} t^{3/2}} + \text{ES as } t \downarrow 0.
\]

Thus

\[
Z(t) = \frac{\pi^{1/2}}{2^{7/2} t^{3/2}} \exp(\delta^2 t) + \text{ES as } t \downarrow 0
\]

(4πt)^{3/2}

(4πt)^{3/2}

\[
\frac{\text{Vol } G}{(4\pi t)^{3/2}} \exp(\delta^2 t) + \text{ES as } t \downarrow 0 \quad \text{by [2].}
\]
4. Rank $G > 1$. In this case we need one auxiliary result.

**Lemma 3.** Let $h_q(x)$ be a homogeneous polynomial of degree $q$ in $\mathbb{R}^n$ with $n > 1$. Then

$$h_q(x) = P_0(x) + |x|^2 P_1(x) + \cdots + |x|^{2d} P_d(x)$$

where $P_i(x)$ is a harmonic homogeneous polynomial of degree $q - 2i$ and $d = \lfloor q/2 \rfloor$.

**Proof.** See [3, p. 139].

Applying Lemma 3 to $f^2(x)$ if Rank $G > 1$, we see $f^2(x) = P_0(x) + |x|^2 P_1(x) + \cdots + c|x|^{2a}$ where $a$ is the degree of $f(x)$ which is the number of positive roots of $\varphi_G$. $P_a(x) = c$ a polynomial of degree 0. Then

$$Z(t) = \frac{\exp|\delta|^2 t}{|w|} \sum_{\Lambda \in L} f^2(\Lambda) \exp(-|\Lambda|^2 t)$$

$$= \frac{\exp|\delta|^2 t}{|w|} \sum_{k=0}^a \sum_{\Lambda \in L} P_k(\Lambda)|\Lambda|^{2k} \exp(-|\Lambda|^2 t).$$

We will show that if $k < a$ then

$$\sum_{\Lambda \in L} P_k(\Lambda)|\Lambda|^{2k} \exp(-|\Lambda|^2 t)$$

is exponentially small as $t \downarrow 0$. To do so, merely observe (3) equals

$$(-1)^k \frac{d^k}{dt^k} \left( \sum_{\Lambda \in L} P_k(\Lambda) \exp(-|\Lambda|^2 t) \right)$$

$$= (-1)^k \frac{d^k}{dt^k} \left( V_L^{-1} t^{2k}(t/\pi)^{-n/2-2k} \sum_{\Lambda \in L'} P_k(\Lambda) \exp(-\pi^2|\Lambda|^2 t) \right).$$

After differentiating we have a finite sum of terms of the form

$$c't^{-\tau} \sum_{\Lambda \in L'} Q_l(\Lambda) \exp(-\pi^2|\Lambda|^2 t)$$

with $Q_l(x)$ homogeneous of degree $l \geq 2a - 2k$. Then $Q_l(0) = 0$ and the sum is exponentially small as $t \downarrow 0$. Thus we need only consider the single term

$$\sum_{\Lambda \in L} c|\Lambda|^{2a} \exp(-|\Lambda|^2 t).$$

This equals

$$(-1)^a \frac{d^a}{dt^a} \left( c \sum_{\Lambda \in L} \exp(-|\Lambda|^2 t) \right) = (-1)^a \frac{d^a}{dt^a} \left( \frac{c \pi^{n/2}}{V_L t^{n/2}} \sum_{\Lambda \in L'} \exp(-|\Lambda|^2 \pi^2 t) \right)$$

$$= \frac{c''}{t^{n/2+a}} \sum_{\Lambda \in L'} \exp(-|\Lambda|^2 \pi^2 t) + ES \text{ as } t \downarrow 0.$$
So

\[ Z(t) = \frac{\exp |\delta|^2 t}{|w|} \frac{c''}{t^{\dim G/2}} + \text{ES} \quad \text{as } t \downarrow 0. \]

However \( Z(t) \sim \frac{(\text{Vol } G)}{(4\pi t)^{\dim G/2}} \) as \( t \downarrow 0 \), so

\[ Z(t) = \frac{\text{Vol } G}{(4\pi t)^{\dim G/2}} \exp |\delta|^2 t + \text{ES} \quad \text{as } t \downarrow 0. \]

**BIBLIOGRAPHY**


Department of Mathematics, University of Miami, Coral Gables, Florida 33124