THE ASYMPTOTIC EXPANSION OF THE ZETA-FUNCTION OF A COMPACT SEMISIMPLE LIE GROUP

ROBERT S. CAHN

Abstract. If \( G \) is a connected, simply connected, semisimple Lie group with metric given by the negative of the Killing form and zeta-function \( Z(t) \), then

\[
Z(t) = \frac{\text{Vol} \ G}{(4\pi t)^{\text{dim} \ G/2}} \exp |\delta|^2 t + \text{exponentially small error as } t \downarrow 0.
\]

0. Introduction. Let \( G \) be a compact, connected, simply connected, semisimple Lie group with Lie algebra \( \mathfrak{g} \). As in [2] the metric \( g \) on \( G \) will be obtained by left translation of the negative of the Killing form of \( \mathfrak{g} \) to the entire group. The Laplacian of \( G \) is then the negative of the Casimir operator and the zeta-function of \((G, g)\) is

\[
Z(t) = \frac{1}{|w|} \sum_{\Lambda \in L} f^2(\Lambda) \exp[-(|\Lambda|^2 - |\delta|^2)t].
\]

In (0), \( f(\Lambda) = \Pi_{\alpha > 0}(\Lambda, \alpha)/\Pi_{\alpha > 0}(\delta, \alpha), \delta = \frac{1}{2} \sum_{\alpha > 0} \alpha, \) \( L \) is the lattice of integral weights, \( |w| \) is the order of the Weyl group and inner products and norms are with respect to the Killing form. We will prove the following

Theorem .

\[
Z(t) = \frac{\text{Vol} \ G}{(4\pi t)^{\text{dim} \ G/2}} \exp |\delta|^2 t + \text{exponentially small error as } t \downarrow 0.
\]

1. Theta relations. In this section we will state some well-known theta relations with minor modifications that are necessary for our purposes. Let \( \mathbb{R}^n \) be Euclidean \( n \)-space endowed with the usual inner product. Then the Laplacian on \( \mathbb{R}^n \), \( \Delta \), is just \( \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2 \) where \( \{x_1, \ldots, x_n\} \) is an orthonormal basis. We assume \( P_q(x) \) is a homogeneous polynomial of degree \( q \) such that \( \Delta P_q \equiv 0 \).

Lemma 1.

\[
\sum_{m \in \mathbb{Z}^n} P_q(m) \exp(-t \pi |m|^2) = i^q t^{-n/2-q} \sum_{m \in \mathbb{Z}^n} P_q(m) \exp(-\pi |m|^2/t).
\]
Proof. This result is found in [1].

If \( P_q \equiv 1 \) we obtain the classical theta relation. It is important to remark that if \( q \geq 1 \) then \( P_q(0) = 0 \) and the right-hand side of (1) is ES (exponentially small) as \( t \downarrow 0 \).

Lemma 1 may be modified by taking a general lattice. If \( L \) is a lattice and \( L' \) is its dual then

**Lemma 2.**

\[
\sum_{m \in L} P_q(m) \exp\left(-\pi t |m|^2 \right) = V_{L'}^{-1} i^q t^{-n/2-q} \sum_{m \in L'} P_q(m) \exp\left(-\pi |m|^2 / t \right)
\]

where \( V_L = \text{volume of a fundamental parallelepiped of } L \).

Proof. Lemma 2 follows directly from the Poisson summation formula.

We now proceed to apply Lemma 2 to the problem at hand.

2. Preliminary steps. We now let \( G \) be a fixed group and \( n \) be the rank of \( G \). We fix a Cartan subalgebra of \( \mathfrak{a}_C = \mathfrak{g} \otimes \mathbb{C} \) and, identify \( \mathbb{R}^n \) with the real span of the roots of \( \mathfrak{g}_C \) in \( \mathfrak{x} \), the complex dual of \( \mathfrak{x} \). \( L \) will be the lattice of integral weights. The natural inner product on \( \mathbb{R}^n \) will be given by the Killing form. With respect to this inner product, we pick an orthonormal basis \( \{x_1, \ldots, x_n\} \) and, as before, \( \Delta = \partial^2 / \partial x_1^2 + \cdots + \partial^2 / \partial x_n^2 \). We now consider 2 cases: \( n = 1 \) and \( n > 1 \).

3. Rank \( G = 1 \). If Rank \( G = 1 \), \( \mathfrak{g} \) is \( A_1 \) and \( f(\Lambda) \) is particularly simple. We may take \( f(\Lambda) = \Lambda \) and \( L = \mathbb{Z} \). Then

\[
Z(t) = \frac{1}{2} \exp|\delta|^2 t \sum_{\Lambda \in \mathbb{Z}} \Lambda^2 \exp(-|\Lambda|^2 t) \quad \text{with } |\Lambda|^2 = 2\Lambda^2.
\]

But

\[
\sum_{\Lambda \in \mathbb{Z}} \Lambda^2 \exp(-2\Lambda^2 t) = -\frac{d}{dt} \left( \frac{1}{2} \sum_{\Lambda \in \mathbb{Z}} \exp(-2\Lambda^2 t) \right)
\]

\[
= -\frac{d}{dt} \left( \frac{\pi^{1/2}}{2^{3/2} t^{1/2}} \sum_{\Lambda \in \mathbb{Z}} \exp(-\Lambda^2 \pi^2 / 2t) \right)
\]

\[
= \frac{\pi^{1/2}}{2^{5/2} t^{3/2}} \sum_{\Lambda \in \mathbb{Z}} \exp(-\pi^2 \Lambda^2 / t) - \frac{\pi^{5/2}}{2^{5/2} t^{5/2}} \sum_{\Lambda \in \mathbb{Z}} \Lambda^2 \exp(-\Lambda^2 \pi^2 / 2t)
\]

\[
= \frac{\pi^{1/2}}{2^{5/2} t^{3/2}} + \text{ES} \quad \text{as } t \downarrow 0.
\]

Thus

\[
Z(t) = \frac{\pi^{1/2}}{2^{7/2} t^{3/2}} \exp|\delta|^2 t + \text{ES} \quad \text{as } t \downarrow 0
\]

\[
\text{Vol } G \exp|\delta|^2 t + \text{ES} \quad \text{as } t \downarrow 0 \text{ (by [2]).}
\]
4. Rank $G > 1$. In this case we need one auxiliary result.

**Lemma 3.** Let $h_q(x)$ be a homogeneous polynomial of degree $q$ in $\mathbb{R}^n$ with $n > 1$. Then

$$h_q(x) = P_0(x) + |x|^2 P_1(x) + \cdots + |x|^{2d} P_d(x)$$

where $P_i(x)$ is a harmonic homogeneous polynomial of degree $q - 2i$ and $d = [q/2]$.

**Proof.** See [3, p. 139].

Applying Lemma 3 to $f^2(x)$ if Rank $G > 1$, we see $f^2(x) = P_0(x) + |x|^2 P_1(x) + \cdots + c|x|^{2a}$ where $a$ is the degree of $f(x)$ which is the number of positive roots of $g_C$. $P_a(x) = c$ a polynomial of degree 0. Then

$$Z(t) = \frac{\exp|\delta|^2 t}{|w|} \sum_{\Lambda \in L} f^2(\Lambda) \exp(-|\Lambda|^2 t)$$

$$= \frac{\exp|\delta|^2 t}{|w|} \sum_{k=0}^{a} \sum_{\Lambda \in L} P_k(\Lambda) |\Lambda|^{2k} \exp(-|\Lambda|^2 t).$$

We will show that if $k < a$ then

$$\sum_{\Lambda \in L} P_k(\Lambda) |\Lambda|^{2k} \exp(-|\Lambda|^2 t)$$

is exponentially small as $t \downarrow 0$. To do so, merely observe (3) equals

$$(-1)^k \frac{d^k}{dt^k} \left( \sum_{\Lambda \in L} P_k(\Lambda) \exp(-|\Lambda|^2 t) \right)$$

$$= (-1)^k \frac{d^k}{dt^k} \left( V_L^{-1} i^{2k} (t/\pi)^{-n/2} \sum_{\Lambda \in L'} P_k(\Lambda) \exp(-\pi^2 |\Lambda|^2/t) \right).$$

After differentiating we have a finite sum of terms of the form

$$c' t^{-r} \sum_{\Lambda \in L'} Q_l(\Lambda) \exp(-\pi^2 |\Lambda|^2/t)$$

with $Q_l(x)$ homogeneous of degree $l \geq 2a - 2k$. Then $Q_l(0) = 0$ and the sum is exponentially small as $t \downarrow 0$. Thus we need only consider the single term

$$\sum_{\Lambda \in L} c |\Lambda|^{2a} \exp(-|\Lambda|^2 t).$$

This equals

$$(-1)^a \frac{d^a}{dt^a} \left( c \sum_{\Lambda \in L} \exp(-|\Lambda|^2 t) \right) = (-1)^a \frac{d^a}{dt^a} \left( \frac{c \pi^{n/2}}{V_L t^{n/2}} \sum_{\Lambda \in L'} \exp(-|\Lambda|^2 \pi^2/t) \right)$$

$$= \frac{c''}{i^{n/2+a}} \sum_{\Lambda \in L'} \exp(-|\Lambda|^2 \pi^2/t) + ES \quad \text{as} \quad t \downarrow 0.$$
So

\[ Z(t) = \frac{\exp |\delta|^2 t}{|w|} \frac{c''}{t^{\dim G/2}} + ES \quad \text{as } t \downarrow 0. \]

However \( Z(t) \sim \frac{\text{Vol } G}{(4\pi t)^{\dim G/2}} \) as \( t \downarrow 0 \), so

\[ Z(t) = \frac{\text{Vol } G}{(4\pi t)^{\dim G/2}} \exp |\delta|^2 t + ES \quad \text{as } t \downarrow 0. \]

BIBLIOGRAPHY


Department of Mathematics, University of Miami, Coral Gables, Florida 33124