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A PROPERTY OF THE PERIODS OF A PRYM DIFFERENTIAL

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ABSTRACT. The periods of Prym differentials can be used to prove the invariance of Picard bundles on Jacobian varieties.

Let S be a compact Riemann surface of genus g with universal covering surface U with the deck transformation group D. For any homomorphism $X\colon D \to \mathbf{C}^* = \mathbf{C} - \{0\}$ into the group of complex units, a (meromorphic) Prym differential on S with multipliers X is a meromorphic differential w on U, satisfying the condition

$$w(du) = X(d)w(u) \text{ for all } d \text{ in } D \text{ and } u \text{ in } U.$$

Let p be a fixed point of U. For all X, consider the space of Prym differentials which are regular except for a pole of at most order one at p or its translates under D. This space has dimension g and varies complex analytically as one varies X (see below). These spaces form a vector bundle over the space of all X. In this paper, I show how to trivialize this vector bundle over a g dimensional real torus, consisting of special X's, by means of the periods of Prym differentials.

In a previous paper [3], I proved that such a trivialization was possible by abstract reasoning. The methods of this paper are more elementary and yield an explicit trivialization. Some basic facts about this vector bundle can be found in [2]. A general reference for periods of Prym differentials is the book [1].

Given any Prym differential w, let V be the largest open subset of U such that w has zero residues everywhere in V. Denote the image of V in S by T.

Let v be any fixed point of V and let t be its image in T. For any path σ in T, beginning and ending at t, the σ period A_σ of w is defined by the integral

$$A_\sigma = -X(\underline{\sigma})^{-1} \int_{\tilde{\sigma}} w,$$

where $\tilde{\sigma}$ is the unique path in U, lifting σ, which begins at t and ends at $\underline{\sigma} \cdot t$. The deck transformation $\underline{\sigma}$ will henceforth be denoted by σ. Thus, A_σ depends only on the homotopy class of σ in the fundamental group of T. Furthermore, A_σ is defined for most σ in any homotopy class. We can then regard A_σ as a function on $\pi_1(T, t)$. Also, it satisfies the cocycle identity,

$$A_{\sigma\tau} = X(\tau)^{-1} A_\sigma + A_\tau \quad \text{for all } \sigma \text{ and } \tau \text{ in } \pi_1(T, t).$$

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Let $[\sigma, \tau]$ be the commutator $\sigma\tau\sigma^{-1}\tau^{-1}$. The cocycle identity implies that

$$(*) \qquad A_{[\sigma,\tau]} = X(\tau)X(\sigma)[(1 - X(\sigma)^{-1})A_\tau - (1 - X(\tau)^{-1})A_\sigma].$$

Let $\sigma_1, \tau_1, \ldots, \sigma_g, \tau_g$ be a standard set [1] of closed arcs, beginning and ending at t, that dissect S. Let R be the area in U bounded by the lifting $\widetilde{[\tau_1, \sigma_1] \cdots [\tau_g, \sigma_g]}$.

PROPOSITION 1. *Let w be a Prym differential with multipliers X. If w has no poles on the boundary of R, then*

$$-2\pi i\left(\sum_{u\in R} \textit{Residue of } w \textit{ at } u\right) = \sum_{1\leqq i\leqq g} A_{[\tau_i,\sigma_i]}.$$

The proof of this proposition is a straightforward application of the residue formula and the cocycle identity.

The other fact that I will need is

PROPOSITION 2. *Let w_1 be a Prym differential with multipliers X and periods A, and let w_2 be a Prym differential with conjugate inverse multipliers $\overline{X}^{-1}$ and periods B. Let w_1 and w_2 have no poles on the boundary of R. Assume that*

(1) w_1 has zero residues everywhere, and

(2) order w_1 + order $w_2 \geqq -1$ *everywhere. Then $\int_R \overline{w}_2 \wedge w_1$ equals the sum*

$$\sum_{1\leqq j\leqq g} \overline{B}_{\tau_j} X(\tau_j)^{-1} A_{\sigma_j} - \overline{B}_{\sigma_j} X(\sigma_j)^{-1} A_{\tau_j}$$
$$- \overline{B}_{\sigma_j} X(\tau_j\sigma_j)^{-1}(A_{\lambda_{j-1}} - X(\tau_j)A_{\lambda_j}) + \overline{B}_{\tau_j} X(\sigma_j\tau_j)^{-1}(A_{\lambda_j} - X(\sigma_j)A_{\lambda_{j-1}})$$

where $\lambda_j = \prod_{1\leqq i\leqq j} [\tau_i, \sigma_i]$.

PROOF. The second assumption assures that the integral $\int_R \overline{w}_2 \wedge w_1$ converges. By the first assumption, we can find a meromorphic function F on U such that $dF = w_1$ and $F(v) = 0$. Then $F(du) = X(d)(F(u) - A_d)$ for all d in D and all u in U. By the calculation in the proof of Theorem 21 in [1], $\int_R \overline{w}_2 \wedge w_1 = -\int_{\tilde{\lambda}_g} w_2 F$ equals the above expression. Q.E.D.

Let p be any fixed point in the interior of R. From the previous two propositions, we can conclude

PROPOSITION 3. *Let w be a Prym differential with multipliers X. Assume w is regular except for a pole of at most order one at p and its translates. Assume that $|X(d)| = 1$ for all d in D. Further, assume that $X(\sigma_i) = 1$ for all $1 \leqq i \leqq g$. Then w is determined by its g periods A_{σ_i}.*

PROOF. As $X(\sigma_i) = 1$, from equation $(*)$, we get $A_{[\tau_i,\sigma_i]} = [X(\tau_i) - 1]A_{\sigma_i}$.

Assume the above differential w has all its σ_i periods equal to zero. Then the $A_{[\tau_i,\sigma_i]}$ are zero by the above equation. By the first proposition, the residue of

w at its only pole in R is zero. Hence, w must be regular at p because its pole has at most order one.

As $X = \overline{X}^{-1}$ by assumption, we may apply the second proposition to compute the integral $\int_R \overline{w} \wedge w$ for the regular differential w. To do this, let us first notice that $A_{\lambda_j} = A_{\lambda_{j-1}} + A_{[\tau_j,\sigma_j]}$. As we know from before that the $A_{[\tau_j,\sigma_j]}$ are zero, we have that the A_{λ_j} are all zero. Thus, the formula in the last proposition reduces to

$$\int_R \overline{w} \wedge w = \sum_{1 \leqq j \leqq g} \overline{A}_{\tau_j} X(\tau_j)^{-1} A_{\sigma_j} - \overline{A}_{\sigma_j} A_{\tau_j}.$$

The second side is zero because the A_{σ_j} are. Therefore, $\int_R \overline{w} \wedge w = 0$, but $\overline{w} \wedge w$ is locally a nonnegative multiple of $\overline{dz} \wedge dz = 2i\,dx \wedge dy$. Hence, we have $\overline{w} \wedge w = 0$. Consequently, $w = 0$.

The proposition follows from the above because the periods are linear functionals on the differentials. Q.E.D.

As a homomorphism X is determined by its $2g$ values $X(\sigma_1), \ldots, X(\sigma_g)$, $X(\tau_1), \cdots, X(\tau_g)$, the set of X's is a complex manifold M isomorphic to $\times_{2g} \mathbf{C}^*$. The sheaf of regular Prym differentials with multipliers X defines a line bundle $\Omega_S(X)$ of degree $2g - 2$ on S. The line bundles $\Omega_S(X)$ depend complex analytically on X. A Prym differential with multipliers X, which is regular except for a pole of order at most one at p and its translates, can be considered as a section of $\Omega_S(X)(q)$, where q is the image of p in S. As $\Omega_S(X)(q)$ has degree $2g - 1$, its space of sections $W(X, q)$ has dimension g by the Riemann-Roch theorem. Furthermore, as X varies, the family $W(X, q)$ forms a complex analytic vector bundle $W(q)$ with base M.

A period A_{σ_i} defines a complex analytic mapping of $W(q)$ to the trivial bundle. If X is one of the systems of multipliers satisfying the conditions of the third proposition, then the g periods $A_{\sigma_1}, \ldots, A_{\sigma_g}$ map $W(X, q)$ isomorphically onto a vector space $\mathbf{C}^g$. As this will consequently happen for all nearby X, the next theorem follows.

THEOREM 1. *There is a maximal open subset N of M, such that*

(1) *N contains all multipliers X with $|X(d)| = 1$ for all d in D and $X(\sigma_i) = 1$ for all $1 \leqq i \leqq g$, and*

(2) *for any X in N, there exists a basis $(w_1(X), \ldots, w_g(X))$ of $W(X, q)$ such that the σ_i period of $w_j(X)$ is one if i equals j and is zero otherwise. Furthermore, the $w_i(X)$ depend complex analytically on X, and form a basis for the vector bundle $W(q)$ restricted to N.*

As noted in the previous paper [3], a theorem of Grauert and the first theorem imply

THEOREM 2. *$W(p)$ is a complex analytically trivial bundle when it is restricted to the manifold of X satisfying $X(\sigma_i) = 1$ for $1 \leqq i \leqq g$.*

Further questions of interest would be: How can one prove the second theorem directly? How large is the open N in the first theorem? What can you say about the τ periods of the normalized Prym differentials $w_i(X)$?

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