A UNIFORMLY CONVEX BANACH SPACE WITH A
SCHAUDER BASIS
WHICH IS SUBSYMMETRIC BUT NOT SYMMETRIC

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Abstract. An example of a uniformly convex Banach space with a basis
\((x_i)\) is constructed such that \((x_i)\) is subsymmetric but not symmetric.

1. Introduction. Garling [3], [4] gave an example of a nonuniformly convex
Banach space with a basis which is subsymmetric but not symmetric. Professor Bill Johnson has conveyed to us that Dacunha Castelle has recently
proved that in \(L_p([0, 1])\), \(1 < p < \infty\), every basic sequence which is subsymmetric is also symmetric. Of course, it is well known that \(L_p([0, 1])\), \(1 < p < \infty\), \(p \neq 2\), has no subsymmetric basis. In this paper, a uniformly convex
Banach space with a basis \((\delta_i)\) is constructed such that \((\delta_i)\) is subsymmetric
but not symmetric. This substantiates a conjecture of William B. Johnson.

We follow the standard notation for Banach space theory. A basis \((x_n)\) for
a space \((X, \|\cdot\|)\) is called unconditionally monotone if
\[ \|\sum a_n x_n\| \leq \|\sum b_n x_n\| \]
whenever \(|a_n| \leq |b_n|\). Suppose \(X\) has an unconditionally monotone basis \((x_i)\).
The norm on \(X\) is called \(p\)-convex if
\[ \|\sum (|a_i|^p + |b_i|^p)^{1/p} x_i\|^p \leq \|\sum |a_i| x_i|^p + \|\sum |b_i| x_i|^p, \quad p > 1, \]
for all scalars \((a_i)\) and \((b_i)\). The norm on \(X\) is said to satisfy an upper
\(l_p\)-estimate if \(\|x + y\|^p \leq \|x\|^p + \|y\|^p\) whenever \(x\) and \(y\) have disjoint support
relative to \((x_i)\). It is clear that if \(\|\cdot\|\) is \(p\)-convex, then it satisfies the upper
\(l_p\)-estimate. The norm on \(X\) is said to satisfy a lower \(l_q\)-estimate, if
\[ \|x + y\|^q \geq \|x\|^q + \|y\|^q\]
whenever \(x\) and \(y\) have disjoint support relative to \((x_i)\). A basis \((x_i)\) is called subsymmetric if it is unconditional and equivalent
to each of its subsequences. It is called symmetric if it is equivalent to each of
its permutations.

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2. The construction of the space. Let \(X\) denote the space of all scalar valued
sequences which have only finitely many nonzero coordinates. As in [4], we
define a norm on \(X\) by
\[ \|(\beta_n)\| = \left( \sup_{n_1 < n_2 < \cdots} \sum \frac{\beta_n^2}{\sqrt{i}} \right)^{1/2}. \]

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It follows immediately that the norm $||\cdot||$ is 2-convex. From now on, let $r = 4, 8$. We define two other norms on $X$ as follows:

$$|(\beta_n)|_r = \left( \sup_{(A_k)} \sum_k \left( \sum_{i \in A_k} \beta_i \delta_i \right)^2 \right)^{1/r},$$

where the $A_k$'s are pairwise disjoint finite subsets of the integers and $(\delta_i)$ are unit vectors. It is a routine calculation to show that $(X, \cdot_r)$ is a Banach space.

**Theorem.** The space $(X, \cdot_r)$ is a uniformly convex Banach space. The unit vectors $(\delta_i)$ constitute a subsymmetric basis for $X$ which is not symmetric.

**Proof.** We divide the proof into several steps.

**Step I.** It is straightforward to show that $(\delta_i)$ is an unconditionally monotone basis for $X$, $(\delta_i)$ is subsymmetric and that $\cdot_r$ satisfies the lower $l_r$-estimate.

**Step II.** We assert that the norm $\cdot_r$ is 2-convex. Indeed, let

$$\sum_i (|\alpha_i|^2 + |\beta_i|^2)^{1/2} \delta_i^2 = \left[ \sum_k \left( \sum_{i \in A_k} (|\alpha_i|^2 + |\beta_i|^2)^{1/2} \delta_i \right)^2 \right]^{2/r}$$

for some pairwise disjoint finite subsets $A_1, A_2, \ldots, A_r$.

$$\leq \left[ \sum_k \left( \left( \sum_{i \in A_k} \alpha_i \delta_i \right)^2 + \left( \sum_{i \in A_k} \beta_i \delta_i \right)^2 \right) \right]^{2/r}$$

as $||\cdot||$ is 2-convex,

$$\leq \left[ \sum_k \left( \left( \sum_{i \in A_k} \alpha_i \delta_i \right)^2 \right)^{r/2} \right]^{2/r} + \left[ \sum_k \left( \left( \sum_{i \in A_k} \beta_i \delta_i \right)^2 \right)^{r/2} \right]^{2/r}$$

as $r = 4, 8$,

$$\leq \sum_i |\alpha_i \delta_i|^2_r + \sum_i |\beta_i \delta_i|^2_r.$$

It follows that $\cdot_r$ is 2-convex.

**Step III.** Since $(X, \cdot_8)$ is 2-convex and satisfies the lower $l_8$-estimate, it is uniformly convex, using a theorem of Figiel and Johnson [2].

**Step IV.** We will now show that $(\delta_i)$ is not symmetric. For $k = 2^m$, $m = 0, 1, 2, \ldots$, define

$$y_m = \left( k^{-1/4}, (k - 1)^{-1/4}, \ldots, 2^{-1/4}, 1, 0, 0, \ldots \right)$$

and
\[ y_m = \left(1, 2^{-1/4}, 3^{-1/4}, \ldots, (k - 1)^{-1/4}, k^{-1/4}, 0, \ldots \right). \]

It is enough [4] to show that

\[ \frac{|y_m|_8}{|y_m|_8} \to \infty \quad \text{as} \quad m \to \infty. \]

Clearly

\[ |y_m|_8 \geq \left(1 + 2^{-1/2} \cdot 2^{-1/2} + \ldots + k^{-1/2} \cdot k^{-1/2}\right)^{1/2} \]

\[ \geq (c_1 + m \cdot \log 2)^{1/2} \]

for large \( m \), \( c_1 \) being a constant.

Now we use \(|\cdot|_4\) to obtain an upper estimate for \(|y_m|_8\). Using the 2-convexity of \(|\cdot|_4\), we get

\[ |y_m|_4^2 \leq \sum_{j=1}^{m+1} |z_j|_4^2 \]

where

\[ z_j = \left(0, 0, \ldots, (2^{m+1-j})^{-1/4}, (2^{m+1-j} - 1)^{-1/4}, \ldots, (2^{m-j} + 1)^{-1/4}, 0, \ldots \right), \quad 1 \leq j \leq m, \]

\[ z_{m+1} = (0, 0, \ldots, 1, 0, \ldots) \]

and

\[ \text{supp}(z_j) \cap \text{supp}(z_l) = \emptyset, \quad j \neq l. \]

To find an upper estimate for \(|z_j|_4\), for the sake of convenience of notation, consider

\[ x_l = \left(0, 0, \ldots, (2l)^{-1/4}, (2l - 1)^{-1/4}, \ldots, (l + 1)^{-1/4}, 0, \ldots \right) \]

where \( l \) is an integer. Let the norm of \( x_l \) be attained over the pairwise disjoint finite subsets \( A_1, A_2, \ldots, A_s \). Let

\[ A_n = \left\{ i_1^{(n)}, i_2^{(n)}, \ldots, i_{r_n}^{(n)} \right\}, \quad i_1^{(1)} = 1, 1 \leq n \leq s. \]

Using an argument as in [4, p. 586] and the fact that \((A_n)_{n=1}^s\) gives the norm for \( x_l \), we have

\[ \sum_{i=1}^{r_n} |r_n| \]

\[ \leq \frac{1}{l + 1} \cdot \sum_{n=1}^{s} \left( \sum_{p=1}^{r_n} \frac{1}{\sqrt{p}} \right)^2 \]

\[ \leq \frac{1}{l + 1} \cdot \sum_{n=1}^{s} 4 \cdot r_n = \frac{4}{l + 1} \cdot l \leq 4 \]

i.e.

\[ |x_l|_4^2 \leq 4 \]
Using (3) in (2), we get

\[(4) \quad |y_m|^4 \leq \sqrt{2(m + 1)} \cdot \]

It can be proved, as in [4], that

\[(5) \quad \|y_m\| \leq c_2 \]

where \(c_2\) is a constant.

Let \((B_i)_{i=1}^s\) be any pairwise disjoint finite subsets of the integers. For the sake of convenience, write \(y_m = \sum \alpha_i \delta_i\). Then we have

\[
\sum_{j=1}^s \left\| \sum_{i \in B_j} \alpha_i \delta_i \right\|^8 \leq \|y_m\|^4 \cdot \sum_{j=1}^s \left\| \sum_{i \in B_j} \alpha_i \delta_i \right\|^4
\]

i.e.

\[
\left( \sum_{j=1}^s \left\| \sum_{i \in B_j} \alpha_i \delta_i \right\|^8 \right)^{1/8} \leq (c_2)^{1/2} \cdot (|y_m|^4)^{1/2}
\]

using (5),

\[
\leq (c_2)^{1/2} (2(m + 1))^{1/4}
\]

using (4).

Since \((B_i)_{i=1}^s\) are arbitrary, we get

\[(6) \quad |y_m|^8 \leq c_3 \cdot (m + 1)^{1/4}. \]

From (1) and (6), we get \(\|y_m\| \rightarrow \infty\) as \(m \rightarrow \infty\). Q.E.D.

**REFERENCES**


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