A UNIFORMLY CONVEX BANACH SPACE WITH A
SCHAUDER BASIS
WHICH IS SUBSYMMETRIC BUT NOT SYMMETRIC

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Abstract. An example of a uniformly convex Banach space with a basis
\((x_i)\) is constructed such that \((x_i)\) is subsymmetric but not symmetric.

1. Introduction. Garling [3], [4] gave an example of a nonuniformly convex
Banach space with a basis which is subsymmetric but not symmetric. Professor Bill Johnson has conveyed to us that Dacunha Castelle has recently
proved that in \(L_p([0, 1]), 1 < p < \infty,\) every basic sequence which is subsymmetric is also symmetric. Of course, it is well known that \(L_p([0, 1]), 1 < p < \infty, p \neq 2,\) has no subsymmetric basis. In this paper, a uniformly convex
Banach space with a basis \((\delta_i)\) is constructed such that \((\delta_i)\) is subsymmetric
but not symmetric. This substantiates a conjecture of William B. Johnson.

We follow the standard notation for Banach space theory. A basis \((x_n)\) for
a space \((X, ||-||)\) is called unconditionally monotone if
\[ ||\sum a_n x_n|| \leq ||\sum b_n x_n|| \]
whenever \(|a_n| \leq |b_n|\). Suppose \(X\) has an unconditionally monotone basis \((x_i)\).
The norm on \(X\) is called \(p\)-convex if
\[ ||\sum (a_i^p + b_i^p)^{1/p} x_i||^p \leq ||\sum a_i x_i||^p + ||\sum b_i x_i||^p, \quad p > 1, \]
for all scalars \((a_i)\) and \((b_i)\). The norm on \(X\) is said to satisfy an upper
\(l_p\)-estimate if \(||x + y||^p \leq ||x||^p + ||y||^p\) whenever \(x\) and \(y\) have disjoint support
relative to \((x_i)\). It is clear that if \(||-||\) is \(p\)-convex, then it satisfies the upper
\(l_p\)-estimate. The norm on \(X\) is said to satisfy a lower \(l_p\)-estimate, if
\(||x + y||^q \geq ||x||^q + ||y||^q\) whenever \(x\) and \(y\) have disjoint support relative to
\((x_i)\). A basis \((x_i)\) is called subsymmetric if it is unconditional and equivalent
to each of its subsequences. It is called symmetric if it is equivalent to each of
its permutations.

The author would like to thank Professor William Johnson and Professor
Ranko Bojanic for several useful conversations regarding this paper.

2. The construction of the space. Let \(X\) denote the space of all scalar valued
sequences which have only finitely many nonzero coordinates. As in [4], we
define a norm on \(X\) by
\[ ||(\beta_n)|| = \left( \sup_{n_1 < n_2 < \ldots} \sum \frac{\beta_n^2}{\sqrt{i}} \right)^{1/2}. \]
It follows immediately that the norm \( \| \cdot \| \) is 2-convex. From now on, let \( r = 4, 8 \). We define two other norms on \( X \) as follows:

\[
|\beta_n|_r = \left( \sup_{(A_k)} \sum_k \left\| \sum_{i \in A_k} \beta_i \delta_i \right\| \right)^{1/r},
\]

where the \( A_k \)'s are pairwise disjoint finite subsets of the integers and \((\delta_i)\) are unit vectors. It is a routine calculation to show that \( (X, |\cdot|_r) \) is a Banach space.

**Theorem.** The space \((X, |\cdot|_8)\) is a uniformly convex Banach space. The unit vectors \((\delta_i)\) constitute a subsymmetric basis for \( X \) which is not symmetric.

**Proof.** We divide the proof into several steps.

**Step I.** It is straightforward to show that \((\delta_i)\) is an unconditionally monotone basis for \( X \), \((\delta_i)\) is subsymmetric and that \(|\cdot|_r\) satisfies the lower \( l_r\)-estimate.

**Step II.** We assert that the norm \(|\cdot|_r\) is 2-convex. Indeed, let

\[
2(K|2 + |A_i|^2)^{1/2} - 2 = 2\sum_{k=1}^j \left\| (\alpha_i^2 + |\beta_i|^2)^{1/2} \delta_i \right\|^{2/r}
\]

for some pairwise disjoint finite subsets \( A_1, A_2, \ldots, A_j \),

\[
\leq \left[ \sum_{k=1}^j \left( \left\| \sum_{i \in A_k} \alpha_i \delta_i \right\|^2 + \left\| \sum_{i \in A_k} \beta_i \delta_i \right\|^2 \right) \right]^{2/r}
\]

as \( \| \cdot \| \) is 2-convex,

\[
\leq \left[ \sum_{k=1}^j \left( \left\| \sum_{i \in A_k} \alpha_i \delta_i \right\|^2 \right)^{r/2} \right]^{2/r} + \left[ \sum_{k=1}^j \left( \left\| \sum_{i \in A_k} \beta_i \delta_i \right\|^2 \right)^{r/2} \right]^{2/r}
\]

as \( r = 4, 8 \),

\[
\leq \left( \sum_{i} |\alpha_i \delta_i|^2 \right)^{r/2} + \left( \sum_{i} |\beta_i \delta_i|^2 \right)^{r/2}.
\]

It follows that \(|\cdot|_r\) is 2-convex.

**Step III.** Since \((X, |\cdot|_8)\) is 2-convex and satisfies the lower \( l_r\)-estimate, it is uniformly convex, using a theorem of Figiel and Johnson [2].

**Step IV.** We will now show that \((\delta_i)\) is not symmetric. For \( k = 2^m \), \( m = 0, 1, 2, \ldots \), define

\[
y_m = (k^{-1/4}, (k - 1)^{-1/4}, \ldots, 2^{-1/4}, 1, 0, 0, \ldots)
\]
\[ \tilde{y}_m = (1, 2^{-1/4}, 3^{-1/4}, \ldots, (k - 1)^{-1/4}, k^{-1/4}, 0, \ldots). \]

It is enough [4] to show that
\[ |\tilde{y}_m|_8/|y_m|_8 \to \infty \quad \text{as} \quad m \to \infty. \]

Clearly
\[
(1) \quad |\tilde{y}_m|_8 \geq (1 + 2^{-1/2} \cdot 2^{-1/2} + \cdots + k^{-1/2} \cdot k^{-1/2})^{1/2}
\geq (c_1 + m \cdot \log 2)^{1/2}
\]
for large \( m \), \( c_1 \) being a constant.

Now we use \( |\cdot|_4 \) to obtain an upper estimate for \( |y_m|_8 \). Using the 2-convexity of \( |\cdot|_4 \), we get
\[
(2) \quad |y_m|_4^2 \leq \sum_{j=1}^{m+1} |z_j|_4^2
\]
where
\[ z_j = (0, 0, \ldots, (2^{m+1-j})^{-1/4}, (2^{m+1-j} - 1)^{-1/4}, \ldots, (2^m - j + 1)^{-1/4}, 0, \ldots), \]
\[ \quad 1 \leq j \leq m, \]
\[ z_{m+1} = (0, 0, \ldots, 1, 0, \ldots) \]
and
\[ \text{supp}(z_j) \cap \text{supp}(z_l) = \emptyset, \quad j \neq l. \]

To find an upper estimate for \( |z_j|_4 \), for the sake of convenience of notation, consider
\[ x_l = (0, 0, \ldots, (2l)^{-1/4}, (2l - 1)^{-1/4}, \ldots, (l + 1)^{-1/4}, 0, \ldots) \]
where \( l \) is an integer. Let the norm of \( x_l \) be attained over the pairwise disjoint finite subsets \( A_1, A_2, \ldots, A_s \). Let
\[ A_n = \{ i_1^{(n)}, i_2^{(n)}, \ldots, i_{r_n}^{(n)} \}, \quad i_1^{(1)} = 1, 1 \leq n \leq s. \]

Using an argument as in [4, p. 586] and the fact that \( (A_n)_{n=1}^s \) gives the norm for \( x_l \), we have \( \sum_{i=1}^s r_i = l \) and
\[
|x_l|_4^4 = \sum_{n=1}^s \left( \sum_{n=1}^{r_n} \frac{1}{\sqrt{2l + 1 - i_p^{(n)}}} \cdot \frac{1}{\sqrt{p}} \right)^2
\leq \frac{1}{l + 1} \cdot \sum_{n=1}^s \left( \sum_{p=1}^{r_n} \frac{1}{\sqrt{p}} \right)^2
\leq \frac{4}{l + 1} \cdot \sum_{n=1}^s 4 \cdot r_n = \frac{4}{l + 1} \cdot l \leq 4
\]
i.e.
\[
|y_m|_4^2 \leq 2.
\]
Using (3) in (2), we get

\[(4) \quad |y_m|_4 \leq \sqrt{2(m + 1)}.\]

It can be proved, as in [4], that

\[(5) \quad \|y_m\| \leq c_2\]

where \(c_2\) is a constant.

Let \((B_i,y_i)_{i=1}^S\) be any pairwise disjoint finite subsets of the integers. For the sake of convenience, write \(y_m = \sum \alpha_i \delta_i\). Then we have

\[
\sum_{j=1}^s \left( \sum_{i \in B_j} \alpha_i \delta_i \right)^8 \leq \|y_m\|_4^4 \sum_{j=1}^s \left( \sum_{i \in B_j} \alpha_i \delta_i \right)^4
\]

i.e.

\[
\left( \sum_{j=1}^s \left( \sum_{i \in B_j} \alpha_i \delta_i \right)^8 \right)^{1/8} \leq (c_2)^{1/2} \cdot (|y_m|_4)^{1/2}
\]

using (5),

\[
\leq (c_2)^{1/2}(2(m + 1))^{1/4}
\]

using (4).

Since \((B_i)_{i=1}^S\) are arbitrary, we get

\[(6) \quad |y_m|_8 \leq c_3 \cdot (m + 1)^{1/4}.
\]

From (1) and (6), we get \(|y_m|_{8/4} = \infty\) as \(m \to \infty\). Q.E.D.

References


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