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FIXED POINTS OF AUTOMORPHISMS

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ABSTRACT. We prove that the fixed point set of a compact Lie group of automorphisms of a Lie group has finitely many components if the Lie group does.

We call a space almost connected if it has only finitely many components. If H is a group acting on a set S let S^H be the set of fixed points of H. In [2] by S. S. Koh it is shown that if G is an almost connected Lie group and A is a finite automorphism group of G, then G^A, the set of fixed points of A, is also almost connected. In this note we prove the following result.

THEOREM. *If G is an almost connected Lie group and A is a compact Lie group acting on G as a Lie transformation group of automorphisms, then G^A is almost connected.*

PROOF. As usual we define the semidirect product $\tilde{G} = A \tilde{\times} G$ by $(\alpha, g) \cdot (\beta, h) = (\alpha\beta, (\beta^{-1}g)h)$ where α, β are in A and g, h are in G. Since A is compact, $\tilde{G}$ is an almost connected Lie group.

There is a compact subgroup K of $\tilde{G}$ and subspaces $E_1, \dots, E_n$ of $\tilde{G}$ satisfying the following condition: $A \subset K$, $E_i = \exp \mathfrak{s}_i$ $(i = 1, \dots, n)$, where $\mathfrak{s}_i$ is a subspace of $\tilde{\mathfrak{g}}$, the Lie algebra of $\tilde{G}$. Here exp is a diffeomorphism and $\mathfrak{s}_i$ is Ad K-invariant. Finally, $\tilde{G} = K \cdot E_1 \cdot \dots \cdot E_n$ (topologically direct) [1, p. 180, Theorem 3.1].

A and G are subgroups of $\tilde{G}$ via the identification of α with (α, e) and g with (I, g) where α is in A and g is in G. (The identities of G and A are denoted by e and I.) We have $\tilde{G}^A = K^A \cdot E_1^A \cdot \dots \cdot E_n^A$ where A-action on $\tilde{G}$ is conjugation. Since K is compact and each $\mathfrak{s}_i$ is linear, $\tilde{G}^A$ is almost connected.

Now we will show G^A is almost connected. Suppose (α, g) is in $\tilde{G}^A$. Then $(\beta, e)(\alpha, g)(\beta, e)^{-1} = (\alpha, g)$ for all β in A. Moreover

$$\begin{aligned}(\beta, e)(\alpha, g)(\beta, e)^{-1} &= (\beta, e)(\alpha, e)(\beta^{-1}, e)(\beta, e)(I, g)(\beta^{-1}, e)\\ &= (\beta\alpha\beta^{-1}, e)(I, \beta(g)) = (\beta\alpha\beta^{-1}, \beta(g)) = (\alpha, g).\end{aligned}$$

Hence $\beta\alpha\beta^{-1} = \alpha$, $\beta(g) = g$ and consequently (α, g) is in $\tilde{G}^A$ if and only if α is in A^A and g is in G^A. Hence $G^A = \Pi_2 \tilde{G}^A$ where Π_2 is the coordinate projection of $A \tilde{\times} G$ onto G.

Therefore G^A is almost connected.

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REMARK. If A is a connected Lie group with simple Lie algebra, then the above result does not hold. Use $A = G =$ the universal covering group of SL(2, R) and let A act by conjugation.

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