A UNIQUENESS RESULT FOR TOPOLOGICAL GROUPS

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Abstract. We give a rapid proof of a general result which has an easy corollary that the p-adic integers have a unique topology in which they are a complete separable metric group.

In [1] Corwin showed that the p-adic integers have a unique topology in which they are a nondiscrete locally compact group. The purpose of this note is to give a rapid proof of the following general theorem. It contains most of Corwin's result as a special case. The proof is by methods different than those employed by Corwin.

Theorem 1. Let G be a complete separable Abelian metric group. For each integer n, let $n \cdot G = \{na | a \in G\}$. Suppose that the translates of the $n \cdot G$ generate the Borel structure of G. Then G has a unique topology in which it is a complete separable metric group.

Proof. It is not a priori obvious that the $n \cdot G$ are Borel subsets of G. However, let P and K be complete separable metric groups, and $\varphi: P \rightarrow K$ a continuous homomorphism. Then $\varphi$ induces a continuous one-to-one homomorphism of $P/\text{kernel } \varphi$ onto $\varphi(P)$. Since $P/\text{kernel } \varphi$ is also a complete separable metric group, Souslin's theorem implies that $\varphi(P)$ is a Borel subset of K. In particular, the $n \cdot G$ are Borel subsets of G.

Let $G'$ be a complete separable metric group which is isomorphic to G as an abstract group but perhaps has a different topology. Let $\varphi: G' \rightarrow G$ be the natural identification. But for each integer n, $n \cdot G' = \varphi^{-1}(n \cdot G)$ is a Borel subset of $G'$. Hence, since the translates of the $n \cdot G$ generate the Borel structure of G, we have that $\varphi$ is a Borel mapping. Hence, by Kuratowski [2, p. 400], there exists a set P of first category in $G'$ such that $\varphi|G' - P$ is continuous.

The proof of the theorem may now be completed in standard fashion. We claim that $\varphi$ is actually continuous on all of $G'$. To show this, let $a_n (n \geq 1)$ and $a$ be elements of $G'$ such that $a_n \rightarrow a$ (as $n \uparrow \infty$). Now if Q is the set which is the union of $a^{-1} \cdot P$ and $a_n^{-1} \cdot P (n \geq 1)$, Q is again a set of the first category. Hence, $G' - Q$ is nonempty. Let b be an element of $G' - Q$. Then $ab$ is in $G' - P$ and $a_n b$ is in $G' - P (n \geq 1)$. But $a_n b \rightarrow ab$. Hence, $\varphi(a_n b) \rightarrow \varphi(ab)$, and so $\varphi(a_n) = \varphi(a_n b) \cdot \varphi(b^{-1}) \rightarrow \varphi(ab) \cdot \varphi(b^{-1}) = \varphi(a)$. Hence, $\varphi$ is a continuous one-to-one mapping of $G'$ onto G. Hence, since both

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$G$ and $G'$ are complete separable metric groups, $\phi$ is actually a topological isomorphism. Q.E.D.

Note that Corwin's result is an immediate corollary. If $G$ is the $p$-adic integers, then the $n \cdot G$ are open subgroups of $G$ which form a basis at the identity. Hence, the translates of the $n \cdot G$ generate the topology and thus the Borel structure of $G$.

**Bibliography**


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