ON ENDMORPHISMS OF A SOLENOID

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ABSTRACT. Geometrically simple Bernoulli generators are constructed for certain ergodic endomorphisms of solenoids. An arbitrary ergodic solenoidal group automorphism is obtained as the limit of a sequence of such Bernoulli factors and hence, by a theorem of D. S. Ornstein, must be measure-theoretically isomorphic to a Bernoulli shift.

In his survey paper [6], B. Weiss stated that, using Y. Katznelson's methods, he can prove that every ergodic automorphism of a solenoid is isomorphic to a Bernoulli shift. The aim of this note is to give an alternative proof of this result, with a partial result in the endomorphism case.

The methods used are similar to those of L. M. Abramov [1], who used geometrically simple generating partitions in order to compute the entropy of certain solenoidal automorphisms. A comparison will show that Abramov's generators are refinements of the Bernoulli generators exhibited in §2 below.

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For brevity, a working knowledge of measure theory, ergodic theory and topological groups is assumed in what follows, apart from the following fundamental definition of a Bernoulli shift.

A measure preserving map \( \phi \) from a separable measure space \((X, \mu)\) to itself will be called a (one-sided) Bernoulli shift if there is a measurable partition \( P \) of \( X \) (called a Bernoulli generator for \( \phi \)) such that

(i) \( \{\phi^{-i}P\} \), \( i \geq 0 \), is an independent family of partitions, and

(ii) \( \bigvee_{i=0}^{\infty} \phi^{-i}P \) is the point partition of \( X \).

If \( \phi \) is invertible, (ii) becomes (ii)' \( \bigvee_{i=0}^{\infty} \phi^{i}P \) is the point partition of \( X \).

1. Details of solenoids are well documented (see e.g. [1], [3] and [2, Chapter VIII]). The following brief characterisation will be subsequently useful.

DEFINITION 1.1. Let \( G \) be a noncyclic subgroup of the discrete additive group \( Q \) of rational numbers. The character group \( \Sigma \) of \( G \), called a (one-dimensional) solenoid, is a compact, separable, commutative topological group.

PROPOSITION 1.2. Let \( a = (a_1, a_2, \ldots) \) be a sequence of integers \( a_i \geq 2 \). Let \( G_a \) be the subgroup of \( Q \) generated by the elements \( \prod_{i=1}^{n} 1/a_i \), for \( n \geq 1 \). Up to isomorphism, every additive subgroup \( G \) of \( Q \), as in 1.1, can be represented as a \( G_a \) for some such \( a \). If \( a \) is the constant sequence on some integer \( a \), then \( G_a \) is the group of \( a \)-ary rationals, denoted \( G_a \).
The character group of $G_a$ will be denoted $\Sigma_a$. The character group of $G_a$ will be denoted $\Sigma_a'$, called the $a$-adic solenoid. The solenoid $\Sigma_a$ can be viewed as a subgroup of the countable product of circle groups $S$, where $x = (x_0, x_1, \ldots) \in \Sigma_a$ if and only if $\forall i \geq 0, x_i \in S$ and $x_{i+1}^a = x_i$.

**Proposition 1.3.** Each endomorphism of $G_a$ is of the form $\psi_{m/n}$, acting as multiplication by $m/n$. By convention, $n > 0$. If $m/n \neq 0$, then $m$ and $n$ are coprime (by convention), and $G_n$ is a subgroup of $G_a$. If $\psi_{m/n}$ is invertible, its inverse is $\psi_{n/m}$, and $G_{mn}$ is a subgroup of $G_a$.

The endomorphisms of $\Sigma_a$ are in 1-1 correspondence with those of $G_a$, so to each endomorphism $\psi_{m/n}$ the dual endomorphism $\phi_{m/n}$ of $\Sigma_a$ is associated. The nontrivial endomorphisms $\psi_{m/n}$ (i.e. $m/n \neq 0$) of $G_a$ are injective, so their duals acting on $\Sigma_a$ are all surjective, Haar measure-preserving endomorphisms.

**Proposition 1.4.** The endomorphism $\phi_{m/n}$ of $\Sigma_a$ is ergodic if and only if $m/n \neq 0, \pm 1$.

2. Let $\Sigma_a$ be a fixed solenoid, and $\phi_{m/n}$ an ergodic endomorphism of it. Up to isomorphism, $\Sigma_a$ may be represented by $a = (nb_1, nb_2, \ldots)$ where each $b_i$ is a positive integer coprime to $n$. Then

$$\phi_{m/n}(x_0, x_1, \ldots) = \phi_{m/n}(x_1^{nb_1}, x_2^{nb_2}, \ldots) = (x_1^{(m/n)nb_1}, x_2^{(m/n)nb_2}, \ldots) = (x_1^{mb_1}, x_2^{mb_2}, \ldots).$$

Before exhibiting Bernoulli partitions for such endomorphisms, some notation and a general lemma must be introduced.

**Notation.** (i) Let $(S, \nu), (\Sigma_a, \mu)$ denote the circle and solenoidal groups, respectively, with normalised Haar measures.

(ii) For each nonzero integer $N$, let $S(N)$ be the partition of $S$ into $|N|$ arcs $\{S_1(N), \ldots, S_{|N|}(N)\}$ where, for $1 \leq j \leq |N|$, $S_j(N) = \{x = \exp 2\pi i \theta: -j - 1/|N| \leq \theta < j/|N|\}$.

Observe that translation by any $N$th root of unity permutes the elements of $S(N)$.

(iii) For $i \geq 0$, let $\pi_i: \Sigma_a \to S$ be the measure-preserving map given by $\pi_i(x_0, x_1, \ldots) = x_i$. 

(iv) For nonzero integers $h$, define $\omega_h: S \to S$ to be the measure preserving endomorphism given by $\omega_h(x) = x^h$.

(v) We shall say that a partition $P$ of a group $X$ is regular with respect to a subgroup $K$ of $X$ if for each atom $P_i \in P$, the sets $\{xP_i\}, x \in K$, are disjoint and form the partition $P$ of $X$.

(vi) For each $r \geq 1$, the product $\prod_{j=1}^r b_j$ will be abbreviated to $B_r$, and by convention $B_0 = 1$.

**Lemma 2.1.** Let $f, g$ be two surjective endomorphisms of a compact separable topological group $X$, with normalised left-invariant Haar measure $\mu$, and let $P$ be a measurable partition of $X$ such that

(i) the elements of $\ker f$ and $\ker g$ commute,

(ii) $g(\ker f) = \ker f$,

(iii) $P$ is regular with respect to $\ker f$.
Then for any measurable partition $Q$ of $X$, the partitions $f^{-1}Q$ and $g^{-1}P$ are independent.

**Proof.** Since $X$ is compact, $\text{Ker} f$ is finite, say of order $p$. The maps $f$ and $g$ must preserve $\mu$. By (iii), $\mu P_i = 1/p$ for $1 \leq i \leq p$.

It follows from (iii) that left translation by any element of $\text{Ker} f$ permutes the elements of $P$, and that the restriction of $f$ to each $P_i$ is a bijection onto $X$.

Let $P_i$, $Q_j$ be elements of $P$, $Q$, respectively. It must be shown that

$$\mu(g^{-1}P_i \cap f^{-1}Q_j) = \mu(g^{-1}P_i) \cdot \mu(f^{-1}Q_j) = (1/p)\mu Q_j.$$  

Let $\tilde{P}$, $\tilde{Q}$ be regular partitions of $g^{-1}P_i$, $f^{-1}Q_j$ with respect to $\text{Ker} f$, $\text{Ker} g$, respectively. (The existence and measurability of such partitions is a simple consequence of Zorn's lemma.)

Then for each atom $\tilde{Q}_j$ of $\tilde{Q}$,

$$f^{-1}Q_j = \bigcup_{x \in \text{Ker} f} x\tilde{Q}_j \quad \text{and} \quad \mu \tilde{Q}_j = (1/p)\mu Q_j.$$  

Thus

$$\mu(g^{-1}P_i \cap f^{-1}Q_j) = \sum_{x \in \text{Ker} f} \mu(g^{-1}P_i \cap x\tilde{Q}_j) = \sum_{x \in \text{Ker} f} \mu(x^{-1} \cdot g^{-1}P_i \cap \tilde{Q}_j).$$

Now

$$\bigcup_{x \in \text{Ker} f} x^{-1} \cdot g^{-1}P_i = \bigcup_{x \in \text{Ker} f} x\left( \bigcup_{y \in \text{Ker} g} y \cdot \tilde{P}_i \right) = \bigcup_{y \in \text{Ker} g} y\left( \bigcup_{x \in \text{Ker} f} x \tilde{P}_i \right) \quad \text{by (i)}$$

where $\tilde{P}_i$ is any atom of $\tilde{P}$. But

$$g\left( \bigcup_{x \in \text{Ker} f} x\tilde{P}_i \right) = \bigcup_{x \in \text{Ker} f} g(x)P_i \quad \text{since} \quad g(\tilde{P}_i) = P_i$$

$$= \bigcup_{x \in \text{Ker} f} xP_i \quad \text{by (ii)}$$

$$= X.$$  

Hence $\bigcup_{x \in \text{Ker} f} x\tilde{P}_i$ contains an atom of a regular partition of $X$ with respect to $\text{Ker} g$, therefore $\bigcup_{x \in \text{Ker} f} x^{-1} \cdot g^{-1}P_i = X$.

But $\mu(g^{-1}P_i) = \mu P_i = 1/p$, so the sets $\{x^{-1} \cdot g^{-1}P_i\}$ are disjoint as $x$ varies over $\text{Ker} f$. Therefore

$$\mu(g^{-1}P_i \cap f^{-1}Q_j) = \mu \tilde{Q}_j = (1/p)\mu Q_j. \quad \text{Q.E.D.}$$

**Proposition 2.2.** Let $\phi_{m/n}$ be an ergodic endomorphism of $\Sigma_{m/n}$, as above. Then for both $N = m$ and $N = n$, the partition $P = \pi_0^{-1} S(N)$ is a Bernoulli partition for $\phi_{m/n}$.
Proof. It will be sufficient to show that for each \( r \geq 0 \), \( \{ \phi_{m/n}^{-i} P \} \), \( 0 \leq i \leq r \), is an independent family of partitions.

Observe

\[
\sqrt[r]{i=0} \phi_{m/n}^{-i} P = \sqrt[r]{i=0} \pi_{-1}^{-1} \omega_{m}^{-i} S(N) = \pi_{-1}^{-1} \omega_{m}^{-i} \omega_{m}^{-r+i} S(N).
\]

By the symmetry of (1), there is no loss of generality in assuming \( N = m \). Then

\[
\sqrt[r]{i=0} \omega_{m}^{-i} \omega_{m}^{-r+i} S(N) = \omega_{m}^{-r} S(m) \vee \omega_{m}^{-1} \sqrt[r]{i=1} \omega_{m}^{-i+1} \omega_{m}^{-r+i} S(m).
\]

In Lemma 2.1 above, set \( f = \omega_{m} \), \( g = \omega_{m} \), \( P = S(m) \) and

\[
Q = \sqrt[r]{i=1} \omega_{m}^{-i+1} \omega_{m}^{-r+i} S(m).
\]

Since \( S \) is commutative, (i) is satisfied. Condition (ii) follows from the coprimeness of \( m \) and \( n' \), and (iii) holds by definition of \( S(m) \). Hence \( \omega_{m}^{-r} S(m) \) and \( \omega_{m}^{-1} \sqrt[r]{i=1} \omega_{m}^{-i+1} \omega_{m}^{-r+i} S(m) \) are independent partitions of \( S \). But \( \pi_{-1} \) and \( \omega_{m} \), preserve measure, so \( P \) and \( \sqrt[r]{i=1} \phi_{m/n}^{-i} P \) are independent. The proof is completed by induction.

The rest of this section concerns the cases in which the Bernoulli partitions obtained above are generators.

**Proposition 2.3.** Let \( \phi_{m/n} \) be an ergodic endomorphism of \( \Sigma_{a} \), with \( |m| > n > 0 \). Then \( P = \pi_{-1}^{-1} S(m) \) is a generator for \( \phi_{m/n} \) if either (i) \( \phi_{m/n} \) is not invertible and \( \Sigma_{a} = \Sigma_{n} \), or (ii) \( \phi_{m/n} \) is invertible and \( \Sigma_{a} = \Sigma_{mn} \).

Proof. (i) Suppose \( \phi_{m/n} \) is not invertible. Let \( a = (nb_{1}, nb_{2}, \ldots) \) where each \( b_{j} \) is coprime to \( n \). Set \( M = |m| \).

(1) Let \( x \) and \( y \) lie in the same atom of \( \sqrt[\infty]{i=0} \phi_{m/n}^{-i} P \). In particular, for each \( r \geq 0 \), \( x_{r} \) and \( y_{r} \) lie in the same atom of \( \omega_{m}^{-1} \sqrt[\infty]{i=0} \omega_{m}^{i} \omega_{m}^{r+i} S(m) \) (as in 2.2(1)).

(2) It can be shown by induction that each atom of \( \sqrt[\infty]{i=0} \omega_{m}^{-i} \omega_{m}^{r+i} S(m) \) \( \vee S(n') \) is contained in a single arc of \( \nu \)-measure at most \( 1/M^{r+1} \).

For \( u, v \in S \), let \( \|u - v\| \) be the \( \nu \)-measure of the shorter arc joining \( u \) and \( v \).

Then by (1) and (2) above, for each integer \( i \) with \( 0 \leq i \leq r \), there is an integer \( s(r,i) \) with \( 0 \leq s(r,i) < B_{r} M^{r+1} \) such that

\[
0 \leq \|x_{r} - y_{r}\| - s(r,i)/B_{r} M^{r+1} - 1/B_{r} M^{r+1}.
\]

Setting \( i = 0 \) in (3) gives \( 0 \leq \|x_{r} - y_{r}\| - s(r,0)/B_{r} M^{r+1} \). But \( x_{r} = y_{r} \) for \( 0 \leq i \leq r - 1 \). Then \( \|x_{r} - y_{r}\| = \alpha/b_{r} n \) for some integer \( \alpha \) with \( 0 \leq \alpha < b_{r} n \). Setting \( i = r \) in (3) gives

\[
0 \leq \alpha/b_{r} n - \alpha/r/b_{r} M^{r+1} - 1/b_{r} M^{r+1}.
\]
Hence $0 \leq B_{r-1} M^r \alpha - n s(r, r) \leq n/M < 1$, which implies $B_{r-1} M^r \alpha = n \cdot s(r, r)$.

But since $B_{r-1}$ and $m$ are coprime to $n$, it follows that $n$ divides $\alpha$. Moreover, if $b_r = 1$, it follows that $\alpha = 0$, i.e. $x_r = y_r$.

Hence, by induction, $x = y$ provided $b_r = 1$, $\forall r \geq 0$ i.e. $\Sigma_a = \Sigma_n$.

(ii) Now suppose $\phi_{m/n}$ is invertible on $\Sigma_a$. Without loss of generality, assume $a = (mnc_1, mnc_2, \ldots)$ where each $c_i$ is an integer coprime to $mn$. Let $M = |m|$ and $C_r = \prod_{j=1}^r c_j$, $C_0 = 1$.

Let $x$ and $y$ be in the same atom of $\bigvee_{r=-\infty}^{\infty} \phi_{m/n} P$. In particular, for each $r \geq 0$, $x_r$ and $y_r$ lie in the same atom of $\omega_{C_r}^{-r} \bigvee_{r=-\infty}^{\infty} \omega_m^{-r-i} \omega_{n-r+i} S(m)$ (compare 2.2(1)).

The atoms of $\bigvee_{r=-\infty}^{\infty} \omega_{C_r}^{-r-i} \omega_{m-r+i} S(m) \vee S(n^{2r})$ are bounded by arcs of measure at most $1/M^{2r+1}$ (compare (2) above). Hence there are integers $t(r, i)$ for $-r \leq i \leq r$ with $0 \leq t(r, i) < C_r M^{r+i}$ such that

$$0 \leq \|x_r - y_r\| - t(r, i)/C_r M^{r+i} n^{-i} \leq 1/C_r M^{2r+1}.$$

(4)

The proof proceeds in the obvious way by analogy with (i), setting $i = 0$, $i = -r$ and $i = +r$ in (4) as required.

The results of this section are now summed up in the following statement.

**Theorem 2.4.** (i) Let $\phi_{m/n}$ be an ergodic ‘ expansive ’ (i.e. $|m/n| > 1$) endomorphism of the $n$-adic solenoid $\Sigma_n$. Then $\phi_{m/n}$ is a one-sided Bernoulli shift.

(ii) The ergodic automorphism $\phi_{m/n}$ of the $mn$-adic solenoid $\Sigma_{mn}$ is a Bernoulli shift.

**Proof.** (i) $P = \pi_0^{-1} S(m)$ is a Bernoulli generator for $\phi_{m/n}$, by 2.2 and 2.3(i).

(ii) If $|m| > n$, then $P = \pi_0^{-1} S(m)$ is a Bernoulli generator for $\phi_{m/n}$, by 2.2 and 2.3(ii). If $|m| < n$, then $Q = \pi_0^{-1} S(n)$ is a Bernoulli generator for $\phi_{n/m}$ by the same argument. But $\phi_{m/n}$ is the inverse of $\phi_{n/m}$, so they share Bernoulli generators.

3. Let $\phi_{m/n}$ now be a fixed but arbitrary ergodic automorphism of a solenoid $\Sigma_a$, and let $M = \max(|m|, n)$. Assume $a = (a_1, a_2, \ldots)$ and set $A_r = \prod_{i=1}^r a_i$.

For $r > 0$, let $P_r = \phi_{a_r} \pi_0^{-1} S(M) = \pi_r^{-1} S(M)$. Now $\phi_{A_r}$ is a $\mu$-preserving endomorphism of $\Sigma_a$, so by 2.2, $P_r$ is a Bernoulli partition for $\phi_{m/n}$. Let $S_r$ be the $\sigma$-algebra generated by the partition $\bigvee_{r=-\infty}^{\infty} \phi_{m/n} P_r$. It follows from the proof of 2.3 (concerning $P_0$, and the definition of $P_r$, that $S_r$ separates points of $\Sigma_a$ having distinct rth coordinates. Hence, $(S_r)$, $r \geq 0$, is a sequence of $\sigma$-algebras increasing to the full $\sigma$-algebra on $\Sigma_a$, and $\phi_{m/n}$ restricted to each $S_r$ is a Bernoulli shift (with Bernoulli generator $P_r$). Hence, the automorphism $\phi_{m/n}$ of $\Sigma_a$ is isomorphic to a generalised Bernoulli shift in the sense of D. S. Ornstein [4], and by his theorem is thus isomorphic to a Bernoulli shift with entropy $\log M$.

**Remark.** Since Ornstein’s theorem only applies in the invertible case, it cannot be deduced by the above method that an arbitrary expansive endomorphism of a solenoid is isomorphic to a one-sided Bernoulli shift.

**References**


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