ON DIRECT SUMS OF REFLEXIVE OPERATORS

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Abstract. Let $A_1$ and $A_2$ be reflexive operators on a Hilbert space $H$. If $A_2$ is algebraic then $A_1 \oplus A_2$ is reflexive.

1. Introduction. Let $H$ be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on $H$. For $A \in B(H)$, $\text{Lat } A$ will denote the set of closed subspaces of $H$ invariant under $A$. If $\mathcal{L}$ is a family of closed subspaces in $H$, then $\text{Alg } \mathcal{L}$ will denote the (weakly closed) algebra of operators which leave invariant every member of $\mathcal{L}$. $\mathcal{U}(A)$ will denote the weak closure of the algebra of polynomials in $A$. $(A)'$ will denote the commutant of $A$.

Definition. $A$ is reflexive if $\mathcal{U}(A) = \text{Alg } \text{Lat } A$. Reflexive operators have been studied by various authors ([1], [2], [4]) and the following question was raised in [3]: If $A_1$ and $A_2$ are reflexive, is $A_1 \oplus A_2$ reflexive?

In this note it is shown that if $A_2$ is algebraic then $A_1 \oplus A_2$ is reflexive.

2. Preliminaries. $H(k)$ will denote the direct sum of $k$ copies of $H$, and for $A \in B(H)$, $A(k)$ will be the $k$-fold copy of $A$ acting on $H(k)$, i.e. if $\langle x_1, \ldots, x_k \rangle \in H(k)$, then $A(k) \langle x_1, \ldots, x_k \rangle = \langle A x_1, \ldots, A x_k \rangle$.

Lemma 1. If for every positive integer $n \leq 1$, $\text{Lat } A_n \subseteq \text{Lat } B_n$, then $B \in \mathcal{U}(A)$.

This lemma is quite well known and is a standard tool used in the study of reflexive operators.

3. Algebraic operators. In this section we obtain some results for algebraic operators. These will be used in the proof of the main theorem.

Theorem 1. Let $A$ be an algebraic operator. Then $\mathcal{U}(A) = (\text{Alg } \text{Lat } A) \cap (A)'$.

Proof. Suppose $B \in (\text{Alg } \text{Lat } A) \cap (A)'$. We will show that for all positive integers $n$, $\text{Lat } A_n \subseteq \text{Lat } B_n$. The result will then follow by Lemma 1.

Suppose $\mathcal{K} \in \text{Lat } A(n)$. Since every invariant subspace of $A(n)$ is a span of cyclic subspaces, we can assume $\mathcal{K}$ is a cyclic subspace. Since $A(n)$ is algebraic, $\mathcal{K}$ is finite dimensional.

For $1 \leq i \leq n$, let $\pi_i(\mathcal{K})$ be the projection of $\mathcal{K}$ on the $i$th coordinate space and let $\pi(\mathcal{K}) = \bigvee_{i=1}^{n} \pi_i(\mathcal{K})$. Then $\pi(\mathcal{K})$ is finite dimensional and is clearly in $\text{Lat } A$ (since $\pi_i(\mathcal{K})$ is, for each $i$). Therefore, $\pi(\mathcal{K}) \in \text{Lat } B$.

Denote the restrictions of $A$ and $B$ to $\pi(\mathcal{K})$ by $\hat{A}$ and $\hat{B}$ respectively. Then
Thus, by [2], there is a polynomial \( p \) such that
\[
Bx = p(A)x \text{ for all } x \in \pi(\mathcal{M}).
\]
Thus if \( \langle x_1, \ldots, x_n \rangle \) is an arbitrary vector in \( \mathcal{M}, x_i \in \pi(\mathcal{M}) \) for \( 1 \leq i \leq n \). Then
\[
Bx_i = p(A)x_i \text{ and } B^{(n)}\langle x_1, \ldots, x_n \rangle = \langle p(A)x_1, \ldots, p(A)x_n \rangle.
\]
This completes the proof.

**Corollary 1.** Let \( A \) and \( B \) be reflexive algebraic operators. Then \( A \oplus B \) is reflexive.

**Proof.** Suppose \( \text{Lat } A \oplus B \subseteq \text{Lat } T \). Since \( \{\langle x, 0 \rangle : x \in H \} \) and \( \{\langle 0, x \rangle : x \in H \} \) are in \( \text{Lat } A \oplus B \), it follows that \( T = E \oplus F \) with \( \text{Lat } A \subseteq \text{Lat } E \) and \( \text{Lat } B \subseteq \text{Lat } F \). Since \( A \) and \( B \) are reflexive, \( AE = EA \) and \( FB = BF \).

Now \( A \oplus B \) is algebraic and \( E \oplus F \in [\text{Alg Lat}(A \oplus B)] \cap (A \oplus B)' \), so we can apply Theorem 1.

**4. The main result.** In this section we show that if \( A_1 \) is algebraic reflexive and \( A_2 \) is reflexive then \( A_1 \oplus A_2 \) is reflexive.

We will first prove a special case and the above result will easily follow.

**Theorem 2.** Suppose \( A \) is nilpotent reflexive and \( B \) is reflexive. Then \( A \oplus B \) is reflexive.

**Proof.** Suppose \( T \in \text{Alg Lat}(A \oplus B) \). The argument used in Corollary 1 shows that \( T = E \oplus F \) with \( E \in \mathcal{U}(A) \) and \( F \in \mathcal{U}(B) \). Thus \( E \oplus F \in (A \oplus B)' \) and \( \text{Lat } A \oplus B \subseteq \text{Lat } E \oplus F \).

It will be shown that \( \text{Lat}(A \oplus B)^{(n)} \subseteq \text{Lat}(E \oplus F)^{(n)} \). It suffices to show this for \( n = 2 \). For then the same hypotheses apply to \( (A \oplus B)^{(2)} \). We can then apply the same argument to show that \( \text{Lat}(A \oplus B)^{(4)} \subseteq \text{Lat}(E \oplus F)^{(4)} \). If we continue this process we obtain that \( \text{Lat}(A \oplus B)^{(2n)} \subseteq \text{Lat}(E \oplus F)^{(2n)} \).

But this implies \( \text{Lat}(A \oplus B)^{(k)} \subseteq \text{Lat}(E \oplus F)^{(k)} \) for all integers \( k \geq 1 \).

We identify \( \text{Lat}(A \oplus B)^{(2)} \) with \( \text{Lat}(A \oplus B)^{(2)}(2) \), and will show
\[
\text{Lat}(A^{(2)} \oplus B^{(2)}) \subseteq \text{Lat}(E^{(2)} \oplus F^{(2)}).
\]

Suppose \( \mathcal{M} \in \text{Lat}(A^{(2)} \oplus B^{(2)}) \). We consider two cases:

**Case (i).** \( \mathcal{M} \) does not contain a vector of the form \( \langle 0, y \rangle \). Here \( 0 \) is the zero vector in \( H^{(2)} \) and \( y \in H^{(2)} \). Then (by a well-known argument) \( \mathcal{M} \) is the graph of some closed operator \( T \) with domain \( \mathcal{D}(T) \) in \( H^{(2)} \); \( \mathcal{M} = \{\langle x, Tx \rangle : x \in \mathcal{D}(T)\} \). Then \( \mathcal{M} \in \text{Lat}(A^{(2)} \oplus B^{(2)}) \) implies \( B^{(2)}Tx = TA^{(2)}x \), \( x \in \mathcal{D}(T) \). Suppose the index of \( A \) is \( n \). Then \( A^{(2)n} = 0 \). This implies that \( B^{(2)n}Tx = 0 \) for \( x \in \mathcal{D}(T) \). If \( \mathcal{M}(L) \) is the null space of the operator \( L \), then \( \mathcal{M} \subseteq \mathcal{M}(A^{(2)n} \oplus B^{(2)n}) = \{\mathcal{M}([A \oplus B]^{(n)})\}^{(2)} \).

Denote the restrictions of \( A \oplus B \) and \( E \oplus F \) to \( \mathcal{M}(A \oplus B)^{(n)} \) by \( S_1 \) and \( S_2 \) respectively. Then \( S_1 \) is nilpotent, and \( S_2 \in (\text{Alg Lat } S_1) \cap (S_1)' \). By Theorem 1, \( S_2 \in \mathcal{U}(S_1) \). This clearly implies that \( \mathcal{M} \in \text{Lat}(E^{(2)} \oplus F^{(2)}) \).

**Case (ii).** \( \mathcal{M} \) contains a vector of the form \( \langle 0, y \rangle \) with \( y \neq 0 \). Let \( \mathcal{M} = \{\langle 0, y \rangle \in \mathcal{M} \} \) and let \( \mathcal{M}' \) denote \( \mathcal{M} \oplus (0) \oplus \mathcal{M} \).

Then \( \mathcal{M}' \) is the graph of some closed operator \( T \); \( \mathcal{M}' = \{\langle x, Tx \rangle : x \in \mathcal{D}(T)\} \). Also...
\[ A^{(2)} \oplus B^{(2)} \langle x, T x \rangle = \langle A^{(2)} x, B^{(2)} T x \rangle = \langle A^{(2)} x, T A^{(2)} x \rangle + \langle 0, (B^{(2)} T - T A^{(2)}) x \rangle, \]

where the last element is in \( \{0\} \oplus \mathcal{N} \).

We can assume that \( \mathcal{N} \) is a cyclic invariant subspace of \( A^{(2)} \oplus B^{(2)} \). Thus there exist vectors \( x_1, x_2, y_1, y_2 \) such that

\[ \mathcal{N} = \bigoplus_{i=0}^{\infty} \langle A^{i} x_1, A^{i} x_2, B^{i} y_1, B^{i} y_2 \rangle. \]

Then if the index of \( A \) on \( \mathcal{N}(T) \) is \( n \), \( \mathcal{N} = \bigoplus_{i=0}^{\infty} \langle B^{i} y_1, B^{i} y_2 \rangle \) and \( \mathcal{N}(T) \) and \( \mathcal{R}(T) \) are finite dimensional. Let \( P \) denote the projection on \( \mathcal{R}(T) \). Since \( \mathcal{N} \in \text{Lat} B^{(2)} \), it follows that \( \mathcal{N}' \in \text{Lat}(A^{(2)} \oplus P B^{(2)} P) \). This implies that \( T A^{(2)} = P B^{(2)} P T \) and therefore \( P B^{(2)} P \) is nilpotent of index \( n \).

Since \( F \in \mathcal{R}(B) \) and \( \mathcal{R}(T) \oplus \mathcal{R} \in \text{Lat} B^{(2)} \), it is easy to see that \( P F(2) P \in \mathcal{R}(P B^{(2)} P) \). Thus there exist polynomials \( q \) and \( r \), both of degree less than \( n \), such that \( E^{(2)} = q(A^{(2)}) \) and \( PF^{(2)} P = r(P B^{(2)} P) \).

To complete the proof it is enough to show that \( \mathcal{N}' \in \text{Lat}(E^{(2)} \oplus P F^{(2)} P) \). This is equivalent to showing that \( r(P B^{(2)} P) = q(P B^{(2)} P) \). Let

\[ \mathcal{N}_{1} = \bigoplus_{i=0}^{\infty} \langle A^{i} x_1, B^{i} y_1 \rangle, \quad \mathcal{N}_{2} = \bigoplus_{i=0}^{\infty} \langle A^{i} x_2, B^{i} y_2 \rangle. \]

Since we identify \( \mathcal{N} \) with the subspace \( \bigoplus_{i=0}^{\infty} \langle A^{i} x_1, B^{i} y_1, A^{i} x_2, B^{i} y_2 \rangle \), then \( \mathcal{N} \subset \mathcal{N}_{1} \oplus \mathcal{N}_{2} \). Since \( A \) is nilpotent, \( \mathcal{N}_{i} = \mathcal{N}_{i} \oplus \{0\} \oplus \mathcal{N}_{i} \) where \( \mathcal{N}_{i} = \bigoplus_{k=0}^{n-1} A^{k} x_1 \) and \( \mathcal{N}_{i}' = \bigoplus_{k=0}^{n-1} B^{k} y_1 \). Then \( \mathcal{R}(S_i) = \bigoplus_{i=0}^{n-1} A^{k} x_1 \) and therefore \( \mathcal{R}(T) \subseteq \mathcal{R}(S_1) \oplus \mathcal{R}(S_2) \). Also \( \mathcal{N} \subseteq \mathcal{N}_{1} \oplus \mathcal{N}_{2} \). By taking orthogonal complements, we obtain that \( \mathcal{R}(S_1) \oplus \mathcal{R}(S_2) \subseteq \mathcal{R}(T) \).

Now \( \mathcal{N}_{i} \in \text{Lat} A \oplus B \) for each \( i \). If \( Q_i \) is the projection on \( \mathcal{R}(S_i) \), it follows that \( S_i A = Q_i B Q_i, S_i \) for \( i = 1, 2 \). Thus \( Q_i B Q_i \) is nilpotent and its index on \( \mathcal{R}(S_i) \) is the index of \( A \) on \( \mathcal{R}(S_i) \). It easily follows that the index of \( (Q_1 \oplus Q_2)(B^{(2)}|Q_1 \oplus Q_2|) \) on \( \mathcal{R}(S_1) \oplus \mathcal{R}(S_2) \) is \( n \). Recall that this is the index of \( P B^{(2)} P \) on \( \mathcal{R}(T) \).

Since \( \mathcal{N}_{i} \in \text{Lat} A \oplus B \subseteq \text{Lat} E \oplus F \), and \( E = q(A) \), it follows that \( Q_i F Q_i = q(Q_i B Q_i) \). Thus

\[ [Q_1 \oplus Q_2] F^{(2)}[Q_1 \oplus Q_2] = q([Q_1 \oplus Q_2] B^{(2)}[Q_1 \oplus Q_2]). \]

Now \( \mathcal{R}(S_1) \oplus \mathcal{R}(S_2) \) is an invariant subspace of \( P B^{(2)} P \). Thus on \( \mathcal{R}(T) = [\mathcal{R}(S_1) \oplus \mathcal{R}(S_2)] \oplus \mathcal{E} \),

\[ PB^{(2)} P = \begin{pmatrix} B_1 & X \\ 0 & B_2 \end{pmatrix} \]

and

\[ PF^{(2)} P = r(PB^{(2)} P) = \begin{pmatrix} r(B_1) & X' \\ 0 & r(B_2) \end{pmatrix}. \]

Thus \( r(B_1) = q(B_1) \). If \( t = r - q \) then \( t(B_1) = 0 \). But the degrees of \( r \) and \( q \)
are less than \( n \) and \( B_1 \) is nilpotent of index \( n \). This implies that \( r = q \). This completes the proof.

**Theorem 3.** Suppose \( A \) is algebraic. If \( A \) and \( B \) are reflexive, then so is \( A \oplus B \).

**Proof.** Since \( A \) is algebraic, \( A \) is similar to an operator \( A' = N_1 \oplus N_2 \oplus \cdots \oplus N_k \) where each \( N_i \) is a translated nilpotent operator with all translating scalars distinct. By the argument used in [2] (which holds equally well in the infinite-dimensional case) \( A' \) is reflexive if and only if \( N_i \) is reflexive for \( 1 \leq i \leq k \). Since reflexivity is preserved under similarities the result follows by induction.

**Bibliography**


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