THE EXACT CARDINALITY OF THE SET OF INVARIANT MEANS ON A GROUP

CHING CHOU

Abstract. The purpose of this note is to show that if G is an infinite amenable group then G has exactly $2^{2^{|G|}}$ invariant means where $|G|$ denotes the cardinality of G.

Let G be a discrete group, $m(G)$ the space of bounded real functions on G with the sup norm. For $x \in G$ and $f \in m(G)$, $l_x f$ and $r_x f$ are defined by $(l_x f)(y) = f(xy)$ and $(r_x f)(y) = f(yx)$, respectively, $y \in G$. $\mu \in m(G)^*$ is called a left (right) invariant mean on $m(G)$ if $\mu > 0$, $\|\mu\| = 1$ and $\mu(l_x f) = \mu(f)$ ($\mu(r_x f) = \mu(f)$) for $x \in G$ and $f \in m(G)$. We shall denote the set of left (right) invariant means on G by $ML(G)$ ($MR(G)$). Let $M(G) = ML(G) \cap MR(G)$, the set of two-sided invariant means, and $M^*(G) = \{ \mu \in M(G): \mu \text{ is inversion invariant} \}$. ($\mu \in m(G)^*$ is said to be inversion invariant if $\mu(f) = \mu(f^*)$ for each $f \in m(G)$ where $f^*$ is defined by $f^*(x) = f(x^{-1})$.) G is said to be amenable if there is at least one left invariant mean on $m(G)$. When G is amenable, $M^*(G) \neq \emptyset$, cf. [3].

If X is a set, $|X|$ will denote the cardinality of X. If G is a finite group then $m(G)$ has a unique left invariant mean $\mu$: $\mu(f) = \frac{1}{|G|} \Sigma \{ f(x): x \in G \}$. E. Granirer [5] proved that this is the only case that $ML(G)$ is a singleton. He actually proved that if G is infinite amenable then the vector space spanned by $ML(G)$ is infinite dimensional. In [1] we showed that if G is infinite amenable then $|ML(G)| \geq 2^c$ where c is the cardinality of the continuum. It implies that the linear span of $ML(G)$ is not even norm separable.

On the other hand, we are unable to find a theorem in the literature stating that $M(G)$ is not a singleton if G is infinite. But if G is assumed to be countably infinite and amenable then we do know that $|M(G)| > 2^c$ [2, Theorem 4.3].

Note that if G is infinite then $|m(G)^*| = 2^{2^{|G|}}$. Since $ML(G) \subseteq m(G)^*$, $|ML(G)| \leq 2^{2^{|G|}}$. Our main result is

Theorem 1. Let G be an infinite amenable group. Then $|ML(G)| = |M(G)| = |M^*(G)| = 2^{2^{|G|}}$.

Clearly it is enough to show that $|M^*(G)| \geq 2^{2^{|G|}}$. In [2, Theorem 4.3] it is

Received by the editors July 11, 1974 and, in revised form, January 9, 1975.


Key words and phrases. Invariant means, amenable groups, cardinalities.

1 Supported in part by NSF grant P034190-001.
proved that if $G$ is countably infinite then there is a one-one mapping of a set \( \mathcal{N} \) of cardinality $2^c = 2^{\aleph_0}$ into $M(G)$. The fact that the sequence \( \{ U_n \} \) in \cite{2} can be chosen to be symmetric implies that the image of \( \mathcal{N} \) is actually contained in $M^*(G)$; see \cite[pp. 450–451]{2}. Therefore it remains to consider the case $|G| > \kappa_0$. We shall actually prove the stronger

**Theorem 2.** Let $G$ be an uncountable amenable group. Then there is a family of subsets \( \{ E_\theta \}_{\theta \in \Theta} \) of $G$, $|\Theta| = 2^{|G|}$, such that each set function on \( \{ E_\theta \}_{\theta \in \Theta} \) with values in $[0,1]$ is the restriction of an element of $M^*(G)$.

Since there are $2^{2^{|G|}}$ such set functions, the uncountable case of Theorem 1 follows directly from the above theorem. To prove Theorem 2 we shall follow closely the steps of the proof of the well-known theorem of Kakutani and Oxtoby \cite{7}. Their theorem is also reproduced in full in Hewitt and Ross \cite[§16]{6}. Therefore we shall skip some of the details. Our notation is similar to that of \cite{6}.

From now on $G$ will denote a fixed amenable group with $|G| > \kappa_0$. Let $T$ be the set of mappings $x \to ax$, $x \to xa$, $x \to x^{-1}$ of $G$ onto itself ($a \in G$). Note that $|T| = |G|$. A set $X \subset G$ is said to be *almost invariant* if $|tA \cap X| < |G|$ for each $t \in T$. The family of almost invariant subsets of $G$ forms an algebra.

**Lemma 1.** There exists a family of subsets \( \{ X_\nu \}_{\nu \in P} \) of subsets of $G$ such that:

(i) $|P| = |G|$;

(ii) the sets $X_\nu$ are mutually disjoint;

(iii) $\bigcup \{ X_\nu : \nu \in P_0 \}$ is almost invariant for each subset $P_0$ of $P$;

(iv) $|X_\nu| = |G|$ for each $\nu \in P$.

**Proof.** Let $\omega$ be the first ordinal number with cardinality $|G|$.

Let \( \{ \tau_a \}_{1 \leq a < \omega} \) be a well ordering of $T$. For $1 \leq a < \omega$, $x \in G$ set $C_a(x) = \{ \tau_{a,1} \circ \cdots \circ \tau_{a,n} x : 1 \leq \beta_k \leq a, \epsilon_k = 1 \text{ or } -1, k = 1, 2, \ldots, n, \text{ and } n = 1, 2, \ldots \}$. Note that $x \in C_a(x)$; if $\beta < a$ then $\tau_{\beta}(C_a(x)) = C_a(x)$ and $|C_a(x)| < |G|$. (The last inequality holds since $|G| > \kappa_0$.) A transfinite double sequence $x^{\alpha,\beta}_v$, $1 \leq \beta < \alpha < \omega$, in $G$ can be constructed such that the sets $C_a(x^{\alpha,\beta}_v)$, $1 \leq \beta < \alpha < \omega$, are mutually disjoint. Let $P = \{ v : v \text{ ordinal number, } 1 \leq v < \omega \}$ and set $X_\nu = \bigcup \{ C_a(x^{\alpha,\beta}_v) : v \leq \alpha < \omega \}$. Then $\{ X_\nu \}_{\nu \in P}$ is what we want. The details can be found on pp. 218–219 of \cite{6}.

**Lemma 2.** There exists a collection of subsets of $P$, \( \{ P_\theta \}_{\theta \in \Theta} \) such that:

(i) $|\Theta| = 2^{|G|}$,

(ii) $\bigcap_{\theta \in \Theta} P_\theta \neq \emptyset$ for each finite sequence $\theta_1, \ldots, \theta_n$ of distinct elements of $\Theta$ and for $\epsilon_1 = 1$ or $'$. (If $A \subset P$, $A^\dagger = A, A' = P \setminus A$.)

In \cite[p. 220]{6} the above lemma is proved for the case that $|P| = |G| = c$. Their proof also works here.

**Proof of Theorem 2.** For each $\theta \in \Theta$, set $E_\theta = \bigcup \{ X_\nu : \nu \in P_\theta \}$. Then, by Lemmas 1 and 2, for each sequence $\theta_1, \ldots, \theta_n$ of distinct elements in $\Theta$ and for $\epsilon_1 = 1$ or $'$, $\bigcap_{\theta \in \Theta} E_{\theta,\epsilon_1}^{\alpha,\beta}$ is almost invariant and with cardinality equal to $|G|$. (Here $E_{\theta}^* = G \setminus E_{\theta}$. Note that for each finite sequence of distinct elements in $\Theta$ the collection \( \{ C_{\alpha} : \nu \in P_\theta, \epsilon = 1 \text{ or } ' \} \) forms a partition of $G$.
into $2^n$ disjoint sets. Let $\mathcal{E}$ be the family of sets consisting of the empty set and sets of the form

$$E = \bigcup \left\{ \bigcap_{k=1}^{n} E_{\theta_k}^* : (\epsilon_1, \ldots, \epsilon_n) \in \Gamma \right\}$$

where $\Gamma$ is a subset of $\{1, \ldots, n\}$, $\theta_1, \ldots, \theta_n$ is a sequence of distinct elements in $\Theta$ and $n = 1, 2, \ldots$. Suppose $E$ is as in (1) and $\theta_1, \ldots, \theta_n, \theta_{n+1}, \ldots, \theta_m$ is a sequence of distinct elements from $\Theta$. Then $E$ can also be written as

$$E = \bigcup \left\{ \bigcap_{k=1}^{m} E_{\theta_k}^* : (\epsilon_1, \ldots, \epsilon_n, \ldots, \epsilon_m) \in \Gamma^- \right\}$$

where $\Gamma^- \subset \{1, \ldots, m\}$ is defined by $(\epsilon_1, \ldots, \epsilon_m) \in \Gamma^-$ if and only if $(\epsilon_1, \ldots, \epsilon_n) \in \Gamma$. Using this remark it is easy to see that $\mathcal{E}$ is an algebra (of almost invariant subsets of $G$).

Denote the set of all functions from $\Theta$ to the closed interval $[0,1]$ by $Z$. Note that $|Z| = 2^{|\Theta|}$. For each $\xi \in Z$ set $\phi_\xi(E_\emptyset) = \xi(\emptyset)$ and $\phi_\xi(E_\emptyset) = 1 - \xi(\emptyset)$. Extend $\phi_\xi$ to $\mathcal{E}$ as follows: $\phi_\xi(\emptyset) = 0$, $\phi_\xi(E) = \Sigma (\prod_{k=1}^n \phi_\xi(E_{\theta_k}^*): (\epsilon_1, \ldots, \epsilon_n) \in \Gamma)$ if $E$ is as in (1). By (2) it is easy to see that $\phi_\xi$ is a well-defined finitely additive measure on $\mathcal{E}$ and $\phi_\emptyset(G) = 1$.

Let $\mathcal{F} = \{F \in G: |F \triangle E| < |G| \text{ for some } E \in \mathcal{E}\}$. Using the fact that each $E \in \mathcal{E}$ is almost invariant one sees:

(3) If $E \in \mathcal{E}$ and $|F \triangle E| < |G|$ then $|\tau F \triangle E| < |G|$ for each $\tau \in T$; therefore $\mathcal{F}$ is closed under $\tau$.

Extend $\phi_\xi$ to $\mathcal{F}$ by setting $\phi_\xi(F) = \phi_\xi(E)$ if $|F \triangle E| < |G|$, $E \in \mathcal{E}$, $\phi_\xi$ is well defined on $\mathcal{F}$ since if $|F \triangle E_1| < |G|$, $|F \triangle E_2| < |G|$, $E_1 \in \mathcal{E}$, then $|E_1 \triangle E_2| < |G|$ and hence $E_1 = E_2$ by (2), Lemma 2(ii) and the definition of $E_\emptyset$. $\phi_\xi$ is clearly finitely additive on $\mathcal{F}$ and, by (3), if $F \in \mathcal{F}$ and $\tau \in T$, $\phi_\xi(\tau F) = \phi_\xi(F)$.

Let $J = \{\sum_{i=1}^k c_i x_{\theta_i}: F_i \in \mathcal{F}, c_i \in R \text{ and } k = 1,2, \ldots \}$ be the vector subspace of $m(G)$ spanned by the characteristic functions $x_{\theta_i}, F \in \mathcal{F}$. If $F \in J$, $x \in G$ then $l_F x, r_F x, f^F$ belong to $J$. Extend $\phi_\xi$ to $J$ in the natural way: $\phi_\xi(\sum_{i=1}^k c_i x_{\theta_i}) = \sum_{i=1}^k c_i \phi_\xi(F_i)$. Then $\phi_\xi$ is a two-sided invariant mean on $J$ which is also inversion invariant.

Now set $K = \{\lambda: \lambda \in m(G)^*, ||\lambda|| = 1, \lambda \geq 0, \lambda|J = \phi_\xi\}$. Then $K$ is a nonempty $\omega^*$-compact convex subset of $m(G)^*$ and $l_F^* K \subseteq K, r_F^* K \subseteq K$ for each $x \in G$. By Day's fixed point theorem [4] there exist $\lambda_\xi, \lambda_\xi^* \in K$ such that $l_F^* \lambda_\xi = \lambda_\xi$ and $r_F^* \lambda_\xi^* = \lambda_\xi^*$, i.e., $\lambda_\xi \in ML(G), \lambda_\xi^* \in MR(G)$ and $\lambda_\xi^*|J = \phi_\xi, \lambda_\xi^*|J = \phi_\xi$. (Note that this is the only place the fact that $G$ is amenable is used.) Define $\lambda_\xi$ as follows: $\lambda_\xi(f) = \lambda_\xi(f^F)$ where $f^F(x) = \lambda_\xi^*(l_F x)$ ($x \in G$). Then $\lambda_\xi \in M(G)$ and $\lambda_\xi|J = \phi_\xi$. Finally set $\mu_\xi(f) = \frac{1}{2} \lambda_\xi(f + f^F), f \in m(G)$. Then $\mu_\xi \in M^*(G), \mu_\xi|J = \phi_\xi$ and, hence, the proof of Theorem 2 is completed.

**Remarks.** 1. The following shorter and more direct proof of Theorem 1 is provided by the referee. Let $G$ be as in Theorem 2. Construct $\{X_{\theta_i}\}_{\theta \in \Theta}$ as in Lemma 1. Let $\eta$ be a finitely additive probability measure on the power sets of $P$. (Since $\eta$ can be considered as a probability measure on $BP$, the
106 CHING CHOU

Stone-Čech compactification of the discrete set $P$, one sees that there are $2^{2^{[G]}}$ such $\eta$'s.) Let $\mathcal{F}$ be the family of all sets of the form $E_Q = \bigcup \{X_\nu : \nu \in Q\}$, $Q \subset P$. Without loss of generality one may assume that $\bigcup \{X_\nu : \nu \in P\} = G$ and, hence, $\mathcal{F}$ is a $\sigma$-algebra of almost invariant subsets of $G$. Now set $\phi(E_Q) = \eta(Q)$. Extend $\phi$ to an element in $M^*(G)$ as in the proof of Theorem 2.

2. In a countably infinite group a set $A$ is almost invariant if and only if $\tau A \triangle A$ is finite for each $\tau \in T$. In the additive group of integers $\mathbb{Z}$ an almost invariant set is either finite or is a complement of a finite set. Therefore Lemma 1 does not hold for $\mathbb{Z}$. On the other hand, it is not hard to see that if $G$ is a countably infinite locally finite group then Lemma 1 holds, e.g., [8, p. 19].

3. For a group $G$, each $\tau \in T$ can be extended to a homeomorphism of $\beta G$ onto $\beta G$ which again will be denoted by $\tau$. A set $K \subset \beta G$ is said to be invariant if $\tau K \subset K$ for each $\tau \in T$. Joe Rosenblatt has recently communicated to us the following fact: If $G$ is an uncountable group, then using the existence of $(E_\theta)_{\theta \in \Theta}$ it can be shown that $\beta G$ contains $2^{2^{[G]}}$ mutually disjoint closed invariant sets. If, in addition, $G$ is amenable, then there exist $2^{2^{[G]}}$ elements in $M^*(G)$ with mutually disjoint supports. The latter fact also follows from Theorem 2. Indeed if $\xi, \zeta : \Theta \to (0,1)$, $\xi \neq \eta$, then there exists $E_\theta$ such that $\mu_\xi(E_\theta) \neq \mu_\zeta(E_\theta)$ and hence support $\mu_\xi \cap$ support $\mu_\eta = \emptyset$. We are unable to decide whether there are $2^c$ mutually singular elements in $M^*(G)$ if $G$ is countably infinite. In [1] we are able to show that $ML(G)$ contains $2^c$ mutually singular elements.

References


Department of Mathematics, State University of New York at Buffalo, Amherst, New York 14226

*2 We wish to thank the referee for this and other useful comments.