THE EXACT CARDINALITY OF THE SET OF INVARIANT MEANS ON A GROUP

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Abstract. The purpose of this note is to show that if $G$ is an infinite amenable group then $G$ has exactly $2^{2^{|G|}}$ invariant means where $|G|$ denotes the cardinality of $G$.

Let $G$ be a discrete group, $m(G)$ the space of bounded real functions on $G$ with the sup norm. For $x \in G$ and $f \in m(G)$, $l_x f$ and $r_x f$ are defined by $(l_x f)(y) = f(xy)$ and $(r_x f)(y) = f(yx)$, respectively, $y \in G$. $\mu \in m(G)^*$ is called a left (right) invariant mean on $m(G)$ if $\mu > 0$, $\|\mu\| = 1$ and $\mu(l_x f) = \mu(f)$ for $x \in G$ and $f \in m(G)$. We shall denote the set of left (right) invariant means on $G$ by $ML(G)$ ($MR(G)$). Let $M(G) = ML(G) \cap MR(G)$, the set of two-sided invariant means, and $M^*(G) = \{ \mu \in M(G) : \mu \text{ is inversion invariant}\}$. ($\mu \in m(G)^*$ is said to be inversion invariant if $\mu(f) = \mu(f^\ast)$ for each $f \in m(G)$ where $f^\ast$ is defined by $f^\ast(x) = f(x^{-1})$.) $G$ is said to be amenable if there is at least one left invariant mean on $m(G)$. When $G$ is amenable, $M^*(G) \neq \emptyset$, cf. [3].

If $X$ is a set, $|X|$ will denote the cardinality of $X$. If $G$ is a finite group then $m(G)$ has a unique left invariant mean $\mu$: $\mu(f) = (1/|G|)\sum \{ f(x) : x \in G \}$. E. Granirer [5] proved that this is the only case that $ML(G)$ is a singleton. He actually proved that if $G$ is infinite amenable then the vector space spanned by $ML(G)$ is infinite dimensional. In [1] we showed that if $G$ is infinite amenable then $|ML(G)| > 2^c$ where $c$ is the cardinality of the continuum. It implies that the linear span of $ML(G)$ is not even norm separable.

On the other hand, we are unable to find a theorem in the literature stating that $M(G)$ is not a singleton if $G$ is infinite. But if $G$ is assumed to be countably infinite and amenable then we do know that $|M(G)| > 2^c$ [2, Theorem 4.3].

Note that if $G$ is infinite then $|m(G)^*| = 2^{2^{|G|}}$. Since $ML(G) \subseteq m(G)^*$, $|ML(G)| < 2^{2^{|G|}}$. Our main result is

**Theorem 1.** Let $G$ be an infinite amenable group. Then $|ML(G)| = |M(G)| = |M^*(G)| = 2^{2^{|G|}}$.

Clearly it is enough to show that $|M^*(G)| > 2^{2^{|G|}}$. In [2, Theorem 4.3] it is
proved that if $G$ is countably infinite then there is a one-one mapping of a set $\mathfrak{N}$ of cardinality $2^c = 2^{\aleph_0}$ into $M(G)$. The fact that the sequence $\{U_n\}$ in [2] can be chosen to be symmetric implies that the image of $\mathfrak{N}$ is actually contained in $M(G)$; see [2, pp. 450–451]. Therefore it remains to consider the case $|G| > \kappa_0$. We shall actually prove the stronger

**Theorem 2.** Let $G$ be an uncountable amenable group. Then there is a family of subsets $\{E_\theta\}_{\theta \in \Theta}$ of $G$, $|\Theta| = 2^{|G|}$, such that each set function on $\{E_\theta\}_{\theta \in \Theta}$ with values in $[0,1]$ is the restriction of an element of $M^*(G)$.

Since there are $2^{2^{|G|}}$ such set functions, the uncountable case of Theorem 1 follows directly from the above theorem. To prove Theorem 2 we shall follow closely the steps of the proof of the well-known theorem of Kakutani and Oxtoby [7]. Their theorem is also reproduced in full in Hewitt and Ross [6, §16]. Therefore we shall skip some of the details. Our notation is similar to that of [6].

From now on $G$ will denote a fixed amenable group with $|G| > \kappa_0$. Let $T$ be the set of mappings $x \rightarrow ax$, $x \rightarrow xa$, $x \rightarrow x^{-1}$ of $G$ onto itself $(a \in G)$. Note that $|T| = |G|$. A set $X \subset G$ is said to be almost invariant if $|\tau A x| < |G|$ for each $\tau \in T$. The family of almost invariant subsets of $G$ forms an algebra.

**Lemma 1.** There exists a family of subsets $\{X_\nu\}_{\nu \in \nu}$ of subsets of $G$ such that:

(i) $|\nu| = |G|$;

(ii) the sets $X_\nu$ are mutually disjoint;

(iii) $\bigcup \{X_\nu: \nu \in \nu_0\}$ is almost invariant for each subset $\nu_0$ of $\nu$;

(iv) $|X_\nu| = |G|$ for each $\nu \in \nu$.

**Proof.** Let $\omega$ be the first ordinal number with cardinality $|G|$. Let $\{\tau_\alpha\}_{1 \leq \alpha < \omega}$ be a well ordering of $T$. For $1 \leq \alpha < \omega$, $x \in G$ set $C_\alpha(x) = C_{\beta_1} \circ \cdots \circ C_{\beta_n} x: 1 \leq \beta_k < \alpha$, $\epsilon_k = 1$ or $-1$, $k = 1, 2, \ldots, n$, and $n = 1, 2, \ldots \}$. Note that $x \in C_\alpha(x)$; if $\beta < \alpha$ then $\tau_\beta(C_\alpha(x)) = C_\alpha(x)$ and $|C_\alpha(x)| < |G|$. (The last inequality holds since $|G| > \kappa_0$.) A transfinite double sequence $X^\alpha_\beta$, $1 \leq \beta < \alpha < \omega$, in $G$ can be constructed such that the sets $C_\alpha(x^\alpha_\beta)$, $1 \leq \beta < \alpha < \omega$, are mutually disjoint. Let $P = \{\nu: \nu \text{ ordinal number, } 1 < \nu < \omega\}$ and set $X_\nu = \bigcup \{C_\alpha(x^\alpha_\nu): 1 < \alpha < \omega\}$. Then $\{X_\nu\}_{\nu \in \nu}$ is what we want. The details can be found on pp. 218–219 of [6].

**Lemma 2.** There exists a collection of subsets of $P$, $\{P_\theta\}_{\theta \in \Theta}$ such that:

(i) $|\Theta| = 2^{|G|},$

(ii) $\bigcap_{1 \leq k \leq n} P_{\theta_k} \neq \emptyset$ for each finite sequence $\theta_1, \ldots, \theta_n$ of distinct elements of $\Theta$ and for $\epsilon_k = 1$ or $'$. (If $A \subset P$, $A' = A$, $A' = P \setminus A$.)

In [6, p. 220] the above lemma is proved for the case that $|P| = |G| = c$. Their proof also works here.

**Proof of Theorem 2.** For each $\theta \in \Theta$, set $E_\theta = \bigcup \{X_\nu: \nu \in P_\theta\}$. Then, by Lemmas 1 and 2, for each sequence $\theta_1, \ldots, \theta_n$ of distinct elements in $\Theta$ and for $\epsilon_k = 1$ or $'$, $\bigcap_{1 \leq k \leq n} E_{\theta_k}^{\epsilon_k}$ is almost invariant and with cardinality equal to $|G|$. (Here $E_{\theta'_\nu} = G \setminus E_{\theta_\nu}$.) Note that for each finite sequence of distinct elements in $\Theta$ the collection $\{C_{\theta_1}^{\epsilon_1} E_{\theta_2}^{\epsilon_2}: 1 < \epsilon_1 < \epsilon_2 < \omega\}$ forms a partition of $G$.
into $2^n$ disjoint sets. Let $\mathcal{E}$ be the family of sets consisting of the empty set and sets of the form

$$E = \bigcup_{k=1}^{n} \left( \bigcap_{i=1}^{k} E_{\theta_i} : (\epsilon_1, \ldots, \epsilon_n) \in \Gamma \right)$$

where $\Gamma$ is a subset of $\{1, \ldots, n\}$, $\theta_1, \ldots, \theta_n$ is a sequence of distinct elements in $\Theta$ and $n = 1, 2, \ldots$. Suppose $E$ is as in (1) and $\theta_1, \ldots, \theta_n, \theta_{n+1}, \ldots, \theta_m$ is a sequence of distinct elements from $\Theta$. Then $E$ can also be written as

$$E = \bigcup_{k=1}^{m} \left( \bigcap_{i=1}^{k} E_{\theta_i} : (\epsilon_1, \ldots, \epsilon_n, \ldots, \epsilon_m) \in \Gamma^- \right)$$

where $\Gamma^- \subseteq \{1, \ldots, n\}$ is defined by $(\epsilon_1, \ldots, \epsilon_m) \in \Gamma^-$ if and only if $(\epsilon_1, \ldots, \epsilon_n) \in \Gamma$. Using this remark it is easy to see that $\mathcal{E}$ is an algebra (of almost invariant subsets of $G$).

Denote the set of all functions from $\Theta$ to the closed interval $[0,1]$ by $Z$. Note that $|Z| = 2^{2\Theta}$. For each $\xi \in Z$ set $\phi_\xi(E_\theta) = \xi(\theta)$ and $\phi_\xi(E_\theta^*) = 1 - \xi(\theta)$. Extend $\phi_\xi$ to $\mathcal{E}$ as follows: $\phi_\xi(\emptyset) = 0$, $\phi_\xi(E) = \sum (\Pi_{i=1}^{\xi} \phi_\xi(E_{\theta_i}^*)) : (\epsilon_1, \ldots, \epsilon_n) \in \Gamma$ if $E$ is as in (1). By (2) it is easy to see that $\phi_\xi$ is a well-defined finitely additive measure on $\mathcal{E}$ and $\phi_\xi(G) = 1$.

Let $\mathcal{F} = \{F \in G : |F \triangle E| < |G| \text{ for some } E \in \mathcal{E}\}$. Using the fact that each $E \in \mathcal{E}$ is almost invariant one sees:

3. If $E \in \mathcal{E}$ and $|F \triangle E| < |G|$ then $|\tau F \triangle E| < |G|$ for each $\tau \in T$; therefore $\mathcal{F}$ is closed under $T$.

4. If $\phi_\xi$ is well defined on $\mathcal{F}$ since if $|F \triangle E_1| < |G|$, $|F \triangle E_2| < |G|$, $E_1 \in \mathcal{E}$, then $|E_1 \triangle E_2| < |G|$ and hence $E_1 = E_2$ by Lemma 2(ii) and the definition of $E_{\theta_1}$. $\phi_\xi$ is clearly finitely additive on $\mathcal{F}$ and, by (3), if $F \in \mathcal{F}$ and $\phi_\xi(F) = \phi_\xi(F)$.

5. Let $J = \{\Sigma_{i=1}^{k} c_i \chi_{F_i} : F_i \in \mathcal{F}, c_i \in R \text{ and } k = 1,2,\ldots\}$ be the vector subspace of $m(G)$ spanned by the characteristic functions $\chi_{F_i}$. If $f \in J$, $x \in G$ then $I_{c_j} f \chi_{F_j} \in J$ belong to $J$. Extend $\phi_\xi$ to $J$ in the natural way:

$$\phi_\xi(\Sigma_{i=1}^{k} c_i \chi_{F_i}) = \Sigma_{i=1}^{k} c_i \phi_\xi(F_i).$$

Then $\phi_\xi$ is a two-sided invariant mean on $J$ which is also inversion invariant.

Now set $K = \{\lambda : \lambda \in m(G)^*, \|\lambda\| = 1, \lambda > 0, \lambda J = \phi_\xi\}$. Then $K$ is a nonempty $\omega$-compact convex subset of $m(G)^*$ and $I_{*} K \subseteq K, r_{*} K \subseteq K$ for each $x \in G$. By Day's fixed point theorem [4] there exist $\lambda_\xi, \lambda_\xi'' \in K$ such that $I_{*} \lambda_\xi = \lambda_\xi$ and $r_{*} \lambda_\xi'' = \lambda_\xi''$, i.e., $\lambda_\xi \in ML(G), \lambda_\xi'' \in MR(G)$ and $\lambda_{\xi J} = \phi_\xi$. (Note that this is the only place the fact that $G$ is amenable is used.) Define $\lambda_{\xi}$ as follows: $\lambda_{\xi}(f) = \lambda_{\xi}(f')$ where $f'(x) = \lambda_{\xi}(f) (x \in G)$. Then $\lambda_{\xi} \in M(G)$ and $\lambda_{\xi} J = \phi_\xi$. Finally set $\mu_\xi(f) = \frac{1}{2} \lambda_{\xi}(f + f), f \in m(G)$. Then $\mu_\xi \in M'(G), \mu_{\xi J} = \phi_\xi$ and, hence, the proof of Theorem 2 is completed.

Remarks. 1. The following shorter and more direct proof of Theorem 1 is provided by the referee. Let $G$ be as in Theorem 2. Construct $(X_{\xi})_{\xi \in \Theta}$ as in Lemma 1. Let $\eta$ be a finitely additive probability measure on the power sets of $P$. (Since $\eta$ can be considered as a probability measure on $BP$, the
Stone-Čech compactification of the discrete set $P$, one sees that there are $2^{2^{[G]}}$ such $\eta$'s.) Let $\mathcal{T}$ be the family of all sets of the form $E_Q = \bigcup \{X_v: v \in Q\}$, $Q \subseteq P$. Without loss of generality one may assume that $\bigcup \{X_v: v \in P\} = G$ and, hence, $\mathcal{T}$ is a $\sigma$-algebra of almost invariant subsets of $G$. Now set $\phi(E_Q) = \eta(Q)$. Extend $\phi$ to an element in $M^*(G)$ as in the proof of Theorem 2.2.

2. In a countably infinite group a set $A$ is almost invariant if and only if $\tau A = A$ is finite for each $\tau \in T$. In the additive group of integers $\mathbb{Z}$ an almost invariant set is either finite or is a complement of a finite set. Therefore Lemma 1 does not hold for $\mathbb{Z}$. On the other hand, it is not hard to see that if $G$ is a countably infinite locally finite group then Lemma 1 holds, e.g., [8, p. 19].

3. For a group $G$, each $\tau \in T$ can be extended to a homeomorphism of $\beta G$ onto $\beta G$ which again will be denoted by $\tau$. A set $K \subseteq \beta G$ is said to be invariant if $\tau K \subseteq K$ for each $\tau \in T$. Joe Rosenblatt has recently communicated to us the following fact: If $G$ is an uncountable group, then using the existence of $\{E_{\theta}\}_{\theta \in \Theta}$ it can be shown that $\beta G$ contains $2^{2^{[G]}}$ mutually disjoint closed invariant sets. If, in addition, $G$ is amenable, then there exist $2^{2^{[G]}}$ elements in $M^*(G)$ with mutually disjoint supports. The latter fact also follows from Theorem 2. Indeed if $\xi, \zeta: \Theta \to (0,1)$, $\xi \neq \eta$, then there exists $E_{\theta}$ such that $\mu_{\xi}(E_{\theta}) \neq \mu_{\zeta}(E_{\theta})$ and hence support $\mu_{\xi} \cap$ support $\mu_{\eta} = \emptyset$. We are unable to decide whether there are $2^c$ mutually singular elements in $M^*(G)$ if $G$ is countably infinite. In [1] we are able to show that $ML(G)$ contains $2^c$ mutually singular elements.

REFERENCES


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2 We wish to thank the referee for this and other useful comments.