THE NUMERICAL RANGE OF AN UNBOUNDED OPERATOR

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Abstract. The numerical range of an unbounded linear operator on a complex Banach space is the whole complex plane.

Let $X$ denote a Banach space over $\mathbb{C}$, $X'$ the dual space of $X$, $S = \{x \in X: \|x\| = 1\}$, and let $T$ be an unbounded operator defined on the whole of $X$. Given $x \in S$, let $V(T,x) = \{f(Tx): f \in X', \|f\| = f(x) = 1\}$. The numerical range $V(T)$ is defined by

$$V(T) = \bigcup\{V(T,x): x \in S\}.$$ 

J. R. Giles and G. Joseph [2] prove that the semi-inner-product numerical range $W(T)$ has a certain density property, and B. Bollobas and S. Eldridge (preprint) prove that $W(T)$ is dense in $\mathbb{C}$. These imply the corresponding results for $V(T)$.

Theorem. $V(T) = \mathbb{C}$.

We use the following slight extension of Theorem 1 of [1].

Lemma. Let $x, y \in X$, and operator $R$ be defined on $\text{lim}(x,y)$. Suppose that $\|x + Ry\| < \|x\| - (8\|Rx\| \|y\|)^{1/2}$. Then $\bigcup\{V(R,z): z \in S \cap \text{lim}(x,y)\}$ contains $0$ as an interior point.

Proof. There is a continuous linear operator $R_1$ on $X$ such that $R = R_1$ on $\text{lim}(x,y)$. The proof in [1] shows that $0$ is an interior point of $\bigcup\{V(R_1,z): z \in S \cap \text{lim}(x,y)\}$ which gives the result.

Proof of Theorem. As in [2], there is a sequence $(x_n)$ in $X$ such that $x_n \to 0$ and $Tx_n \to -x \neq 0$. Choose $x_n$ such that

$$\|x + Tx_n\| < \|x\| - (8\|Tx\| \|x_n\|)^{1/2}.$$ 

By the Lemma $0 \in V(T)$. For any $\alpha \in \mathbb{C}$, $T - \alpha I$ is unbounded, so $0 \in V(T - \alpha I)$. Hence $\alpha \in V(T)$.

The Lemma implies that, for $T$ defined on a subspace of $X$ with $V(T) \subset \mathbb{R}$, where we take $V(T) = \bigcup\{V(T,x): x \in S, Tx \text{ defined}\}$, we have $\|Tx\|^2 \leq M\|x\| \|T^2x\|$ with $M = 8$. A result of Hille [3] implies that this holds with $M = 2$, and an example of Kolmogorov [4] (differentiation on $L_\infty(\mathbb{R})$) shows that 2 is the best constant.

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REFERENCES


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