ON A THEOREM OF BRICKMAN-FILLMORE

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Abstract. Let $V$ be a finite dimensional vector space over an arbitrary field. We show that if $\dim V \leq 3$ and if $A$, $B$ and $C$ are pairwise commuting linear transformations on $V$ such that every subspace invariant for both $A$ and $B$ is also invariant for $C$, then $C$ is a polynomial in $A$ and $B$. (Brickman and Fillmore proved that if $B = 0$ then this statement is true for any finite dimensional vector space $V$.) An example shows that this is not true for $\dim V > 3$.

In [1] L. Brickman and P. A. Fillmore proved that if $A$ and $B$ are commuting linear transformations on a finite dimensional vector space over an arbitrary field and if every subspace invariant for $A$ is also invariant for $B$, then $B$ is a polynomial in $A$. Peter Fillmore suggested the following question (conveyed to me by Constantin Apostol):

If $A$, $B$ and $C$ are pairwise commuting linear transformations on a finite dimensional vector space $V$ over an arbitrary field and if every subspace invariant for both $A$ and $B$ is also invariant for $C$, then is $C$ a polynomial in $A$ and $B$?

We shall prove that the answer to this question is true if the dimension of $V$ is not more than 3 and false otherwise.

Suppose the dimension of $V$ is 2. If $A$ has no nontrivial invariant subspace then $C$ is a polynomial in $A$ by the Brickman-Fillmore result. If $A$ is a scalar multiple of the identity then $C$ is a polynomial in $B$. Similar statements can also be made for $B$. Finally, if $A$ has a 1-dimensional eigenspace then $A$, $B$ and $C$ can be represented by upper triangular matrices relative to a fixed basis. By subtracting appropriate scalar multiples of the identity from $A$, $B$, and $C$, we may assume that:

$$
A = \begin{pmatrix} 0 & a_1 \\ 0 & a_2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b_1 \\ 0 & b_2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & c_1 \\ 0 & c_2 \end{pmatrix}.
$$

Since $A$ and $C$ commute we have $a_1 c_2 = c_1 a_2$. Thus (i) $a_1 \neq 0$ implies $(c_1/a_1)A = C$, (ii) $a_2 \neq 0$ implies $(c_2/a_2)A = C$ and (iii) $a_1 = a_2 = 0$ implies $C$ is a polynomial in $B$.

The proof for the case when the dimension of $V$ is 3 is obtained by considering the possible representations of $A$ given by the rational decomposition theorem. We omit the details.

Finally, let

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An easy computation shows that

\[ AB = BA = AC = CA = BC = CB = 0 \quad \text{and} \quad A^2 = B^2 = C^2 = 0. \]

It follows from these that \( C \) is not a polynomial in \( A \) and \( B \). To show that every subspace invariant under \( A \) and \( B \) is also invariant under \( C \) it is sufficient to consider cyclic subspaces (that is, subspaces generated by the action of \( A \) and \( B \) on a single vector). An easy calculation shows that if \( x \) is any vector, then \( Cx \) is a linear combination of \( Ax \) and \( Bx \). This example can be extended to the case \( \dim V > 4 \) via direct sums.

**Reference**