A FIXED POINT THEOREM FOR A SYSTEM OF MULTIVALUED TRANSFORMATIONS

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Abstract. We shall prove a fixed point theorem for a system of multivalued mappings which generalizes the result obtained by the author [1, Theorem 1]. For \( n = 1 \) we obtain a generalization of results of Reich [5, Theorem 5] and Nadler [3, Theorem 5], [4, Theorem 1].

1. Let \((X, d)\) be a metric space. We follow the notation of [4].
   (a) \( CL(X) = \{ C : C \) is a nonempty closed subset of \( X \} \),
   (b) \( N(\epsilon, C) = \{ x \in X : d(x, c) < \epsilon \quad \text{for some} \quad c \in C \} , \epsilon > 0 , C \in CL(X) \),
   \( H(A, B) = \begin{cases} 
   \inf\{ \epsilon > 0 : A \subset N(\epsilon, B) \text{ and } B \subset N(\epsilon, A) \}, & \text{if the infimum exists,} \\
   \infty, & \text{otherwise,}
   \end{cases} \quad A, B \in CL(X) \).

   The function \( H \) is called the generalized Hausdorff distance for \( CL(X) \) induced by \( d \). \( D(x, A) \) will denote the ordinary distance between \( x \in X \) and \( A \in CL(X) \).

2. We follow the notation of [2].

\[
\begin{align*}
   c_{i,k}^1 & = \begin{cases} 
   c_{i,k} & \text{for } i \neq k, \\
   1 - c_{i,k} & \text{for } i = k,
   \end{cases} \\
   c_{i,k}^{s+1} & = \begin{cases} 
   c_{i,1,k}^s c_{i,1,k+1}^s + c_{i,1,k}^s c_{i,1,k}^s & \text{for } i \neq k, \\
   c_{i,1,k}^s c_{i,1,k+1}^s - c_{i,1,k}^s c_{i,1,k}^s & \text{for } i = k,
   \end{cases}
\end{align*}
\]

(2)

\( s = 1, \ldots , n - 1, i,k = 1, \ldots, n - s \).

The following result is contained in [2].

**Lemma.** Let \( c_{i,k}^1 > 0, i, k = 1, \ldots, n \). The system of inequalities

\[
\sum_{k=1}^{n} c_{i,k} r_k < r_i, \quad i = 1, \ldots, n,
\]

has a solution \( r_i > 0, i = 1, \ldots, n \), if and only if the following inequalities hold:
Suppose that \( r_i > 0, i = 1, \ldots, n \), is the solution of the system of inequalities (3). We define

\[
\nu = \max_i \left( r_i^{-1} \sum_{k=1}^{n} c_{i,k} r_k \right).
\]

In view of the homogeneity of the system of inequalities (3), definition (5) is correct and

\[
0 < \nu < 1.
\]

Let \( c \) be a real number such that

\[
0 < c < 1 - \nu.
\]

Let \( (X_i, d_i), i = 1, \ldots, n \), be metric spaces. \( H_i(A,B), i = 1, \ldots, n, \) will denote the Hausdorff distance between two elements of \( CL(X_i), i = 1, \ldots, n \), obtained from \( d_i, i = 1, \ldots, n \), and \( D_j(x,A) \) will denote the ordinary distance between \( x \in X_j, A \in CL(X_i) \).

Now we shall prove the following

**Theorem.** Let \( (X_i, d_i), i = 1, \ldots, n \), be complete metric spaces and let \( a_{i,k} > 0, b_{i,k} > 0 \) for \( i, k = 1, \ldots, n \). Let \( c_{i,k} = a_{i,k} + b_{i,k}, i, k = 1, \ldots, n \), be positive and let the numbers \( c_{i,k}^s, s = 1, \ldots, n, i, k = 1, \ldots, n + 1 - s \), defined by (1) and (2) fulfil the inequalities (4). Suppose that the transformations \( F_i: X_1 \times \cdots \times X_n \to CL(X_i), i = 1, \ldots, n \), fulfil

\[
H_i\left( F_i(x_1, \ldots, x_n), F_i(z_1, \ldots, z_n) \right) \leq \sum_{k=1}^{n} a_{i,k} d_k(x_k, z_k)
\]

\[
+ \sum_{k=1}^{n} b_{i,k} D_k(x_k, F_k(x_1, \ldots, x_n))
\]

\[
+ cD_j(z_j, F_j(z_1, \ldots, z_n))
\]

for all \( x_j, z_j \in X_j, i, j = 1, \ldots, n \), where \( c \) fulfils (7). Then the system \( (F_1, \ldots, F_n) \) has a fixed point, i.e. there exist points \( u_i \in X_i, i = 1, \ldots, n \), such that \( u_i \in F_i(u_1, \ldots, u_n) \) for all \( i = 1, \ldots, n \).

**Proof.** Let \( x_i^0 \in X_i, i = 1, \ldots, n \), and choose \( x_i^1 \in F_i(x_1^0, \ldots, x_n^0), i = 1, \ldots, n \). From (1), (2), (4), the Lemma and (5) we may choose a system of positive numbers \( r_1, \ldots, r_n \) such that

\[
\sum_{k=1}^{n} c_{i,k} r_k < \nu r_i, \quad i = 1, \ldots, n.
\]

We may assume (from the homogeneity of the above system) that

\[
d_i(x_i^0, x_i^1) < r_i \quad \text{and} \quad r_i \geq 1 \quad \text{for} \quad i = 1, \ldots, n.
\]

Let \( A, B \in CL(X_i) \) and let \( a \in A \). By definition, if \( q > 0 \), then there exists \( b \in B \) such that \( d(a,b) < H_i(A,B) + q \). Hence in view of conditions \( F_i(x_1, \ldots, x_n), F_i(x_1^i, \ldots, x_n^i) \in CL(X_i) \) and \( x_i^1 \in F_i(x_1^0, \ldots, x_n^0), i = 1, \ldots, n \), there exist points \( x_i^2 \in F_i(x_1^1, \ldots, x_n^1), i = 1, \ldots, n \), such that
By induction, we obtain the sequences \( \{x_i^k\}_{k=1}^\infty, \) \( i = 1, \ldots, n, \) of points of \( X_i, \) \( i = 1, \ldots, n, \) such that \( x_i^k \in F_i(x_i^{k-1}, \ldots, x_i^{k-1}), \) \( i = 1, \ldots, n, \) \( k = 1, 2, \ldots, \) and

\[
d_i(x_i^k, x_i^{k+1}) \leq H_i\left[F_i(x_i^{k-1}, \ldots, x_i^{k-1}), F_i(x_i^k, \ldots, x_i^n)\right] + \nu,
\]

from (11), (8), (10), (9) and the induction principle, we obtain

\[
d_i(x_i^1, x_i^2) \leq \sum_{k=1}^n a_{i,k}d_k(x_i^0, x_i^1) + \sum_{k=1}^n b_{i,k}D_k[x_i^0, F_k(x_i^1, \ldots, x_i^n)] \]
\[
+ cD_k[x_i^1, F_i(x_i^1, \ldots, x_i^n)] + \nu
\]
\[
\leq \sum_{k=1}^n (a_{i,k} + b_{i,k})d_k(x_i^0, x_i^1) + cd_i(x_i^1, x_i^2) + \nu
\]
\[
\leq \sum_{k=1}^n c_{i,k}r_k + cd_i(x_i^1, x_i^2) + \nu
\]
\[
\leq \nu r_i + cd_i(x_i^1, x_i^2) + \nu.
\]

Thus

\[
d_i(x_i^1, x_i^2) \leq \frac{\nu}{1-c} r_i + \frac{\nu}{1-c} \leq 2 \frac{\nu}{1-c} r_i.
\]

Recalling (12), (8), (10), (9) and the induction principle, we obtain

\[
d_i(x_i^k, x_i^{k+1}) \leq (k+1)(\nu/(1-c))r_i, \quad i = 1, \ldots, n, \quad k = 1, 2, \ldots.
\]

Now we have

\[
d_i(x_i^k, x_i^{k+m}) \leq d_i(x_i^k, x_i^{k+1}) + \cdots + d_i(x_i^{k+m-1}, x_i^{k+m})
\]
\[
\leq (k+1)\left(\frac{\nu}{1-c}\right)^k r_i + \cdots + (k+m)\left(\frac{\nu}{1-c}\right)^{k+m-1} r_i
\]
\[
\leq \sum_{s=k}^\infty (s+1)\alpha^s r_i \leq (k+2)\alpha^k (1-\alpha)^{-2} r_i,
\]

for all \( k, m \geq 1, \) \( i = 1, \ldots, n, \) where \( \alpha = \nu/(1-c). \) Thus \( \{x_i^k\}_{k=1}^\infty, \) \( i = 1, \ldots, n, \) are Cauchy sequences and, therefore, \( x_i^k \to u_i \in X_i, \) \( i = 1, \ldots, n. \) We claim that \( (u_1, \ldots, u_n) \) is a fixed point of \( (F_1, \ldots, F_n). \)

Actually

\[
D_i[u_i, F_i(u_1, \ldots, u_n)] \leq d_i(u_i, x_i^{k+1}) + D_i[x_i^{k+1}, F_i(u_1, \ldots, u_n)]
\]
\[
\leq d_i(u_i, x_i^{k+1}) + H_i[F_i(x_i^k, \ldots, x_i^n), F_i(u_1, \ldots, u_n)]
\]
\[
\leq d_i(u_i, x_i^{k+1}) + \sum_{s=1}^n a_{i,s}d_s(x_i^k, u_s)
\]
\begin{align*}
&+ \sum_{s=1}^{n} b_{i,s} D_s \left[ x_s^k, F_s(x_1^k, \ldots, x_n^k) \right] + c D_i \left[ u_i, F_i(u_1, \ldots, u_n) \right] \\
&\leq d_i(u_i, x_i^{k+1}) + \sum_{s=1}^{n} a_{i,s} d_s(x_s^k, u_s) \\
&\quad + \sum_{s=1}^{n} b_{i,s} d_s(x_s^k, x_s^{k+1}) + c D_i \left[ u_i, F_i(u_1, \ldots, u_n) \right].
\end{align*}

Hence,

\begin{align*}
D_i \left[ u_i, F_i(u_1, \ldots, u_n) \right] \\
&\leq \frac{1}{1 - c} \left[ d_i(u_i, x_i^{k+1}) + \sum_{s=1}^{n} a_{i,s} d_s(x_s^k, u_s) \\
&\quad + \sum_{s=1}^{n} b_{i,s} d_s(x_s^k, x_s^{k+1}) \right] \to 0 \quad \text{as } k \to \infty.
\end{align*}

Since \( F_i(u_1, \ldots, u_n) \) is closed, this means that \( u_i \in F_i(u_1, \ldots, u_n) \), \( i = 1, \ldots, n \), which completes the proof.

\textbf{References}


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