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A FIXED POINT THEOREM FOR A SYSTEM OF MULTIVALUED TRANSFORMATIONS

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ABSTRACT. We shall prove a fixed point theorem for a system of multivalued mappings which generalizes the result obtained by the author [1, Theorem 1]. For $n = 1$ we obtain a generalization of results of Reich [5, Theorem 5] and Nadler [3, Theorem 5], [4, Theorem 1].

1. Let (X, d) be a metric space. We follow the notation of [4].

(a) $CL(X) = \{C\colon C$ is a nonempty closed subset of $X\}$,

(b) $N(\varepsilon, C) = \{x \in X\colon d(x, c) < \varepsilon$ for some $c \in C\}$, $\varepsilon > 0$, $C \in CL(X)$,

$$H(A, B) \tag{c}$$
$$= \begin{cases} \inf\{\varepsilon > 0\colon A \subset N(\varepsilon, B) \text{ and } B \subset N(\varepsilon, A)\}, & \text{if the infimum exists,} \\ \infty, & \text{otherwise,} \end{cases}$$
$$A, B \in CL(X).$$

The function H is called the generalized Hausdorff distance for $CL(X)$ induced by d. $D(x, A)$ will denote the ordinary distance between $x \in X$ and $A \in CL(X)$.

2. We follow the notation of [2].

$$c_{i,k}^1 = \begin{cases} c_{i,k} & \text{for } i \neq k, \\ 1 - c_{i,k} & \text{for } i = k, \end{cases} \qquad i, k = 1, \ldots, n, \tag{1}$$

$$c_{i,k}^{s+1} = \begin{cases} c_{1,1}^s c_{i+1,k+1}^s + c_{i+1,1}^s c_{i,k+1}^s & \text{for } i \neq k, \\ c_{1,1}^s c_{i+1,k+1}^s - c_{i+1,1}^s c_{i,k+1}^s & \text{for } i = k, \end{cases} \tag{2}$$
$$s = 1, \ldots, n-1, \; i,k = 1, \ldots, n-s.$$

The following result is contained in [2].

LEMMA. *Let $c_{i,k}^1 > 0$, $i, k = 1, \ldots, n$. The system of inequalities*

$$\sum_{k=1}^{n} c_{i,k} r_k < r_i, \qquad i = 1, \ldots, n, \tag{3}$$

has a solution $r_i > 0$, $i = 1, \ldots, n$, if and only if the following inequalities hold:

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$$c_{i,i}^{s} > 0, \qquad s = 1, \ldots, n, \quad i = 1, \ldots, n + 1 - s. \tag{4}$$

Suppose that $r_i > 0$, $i = 1, \ldots, n$, is the solution of the system of inequalities (3). We define

$$\nu = \max_i \left(r_i^{-1} \sum_{k=1}^{n} c_{i,k} r_k \right). \tag{5}$$

In view of the homogeneity of the system of inequalities (3), definition (5) is correct and

$$0 < \nu < 1. \tag{6}$$

Let c be a real number such that

$$0 \leqslant c < 1 - \nu. \tag{7}$$

Let (X_i, d_i), $i = 1, \ldots, n$, be metric spaces. $H_i(A,B)$, $i = 1, \ldots, n$, will denote the Hausdorff distance between two elements of $CL(X_i)$, $i = 1, \ldots, n$, obtained from d_i, $i = 1, \ldots, n$, and $D_i(x,A)$ will denote the ordinary distance between $x \in X_i$, $A \in CL(X_i)$, $i = 1, \ldots, n$.

Now we shall prove the following

THEOREM. *Let (X_i, d_i), $i = 1, \ldots, n$, be complete metric spaces and let $a_{i,k} \geqslant 0$, $b_{i,k} \geqslant 0$ for $i, k = 1, \ldots, n$. Let $c_{i,k} = a_{i,k} + b_{i,k}$, $i, k = 1, \ldots, n$, be positive and let the numbers $c_{i,k}^{s}$, $s = 1, \ldots, n$, $i, k = 1, \ldots, n + 1 - s$, defined by* (1) *and* (2) *fulfil the inequalities* (4). *Suppose that the transformations F_i: $X_1 \times \cdots \times X_n \to CL(X_i)$, $i = 1, \ldots, n$, fulfil*

$$\begin{aligned} H_i[F_i(x_1, \ldots, x_n), F_i(z_1, \ldots, z_n)] &\leqslant \sum_{k=1}^{n} a_{i,k} d_k(x_k, z_k) \\ &+ \sum_{k=1}^{n} b_{i,k} D_k[x_k, F_k(x_1, \ldots, x_n)] \\ &+ c D_i[z_i, F_i(z_1, \ldots, z_n)] \end{aligned} \tag{8}$$

for all x_j, $z_j \in X_j$, $i,j = 1, \ldots, n$, where c fulfils (7). *Then the system $(F_1, \ldots, F_n)$ has a fixed point, i.e. there exist points $u_i \in X_i$, $i = 1, \ldots, n$, such that $u_i \in F_i(u_1, \ldots, u_n)$ for all $i = 1, \ldots, n$.*

PROOF. Let $x_i^0 \in X_i$, $i = 1, \ldots, n$, and choose $x_i^1 \in F_i(x_1^0, \ldots, x_n^0)$, $i = 1, \ldots, n$. From (1), (2), (4), the Lemma and (5) we may choose a system of positive numbers $r_1, \ldots, r_n$ such that

$$\sum_{k=1}^{n} c_{i,k} r_k \leqslant \nu r_i, \qquad i = 1, \ldots, n. \tag{9}$$

We may assume (from the homogeneity of the above system) that

$$d_i(x_i^0, x_i^1) \leqslant r_i \quad \text{and} \quad r_i \geqslant 1 \quad \text{for } i = 1, \ldots, n. \tag{10}$$

Let $A, B \in CL(X_i)$ and let $a \in A$. By definition, if $q > 0$, then there exists $b \in B$ such that $d_i(a,b) \leqslant H_i(A,B) + q$. Hence in view of conditions $F_i(x_1^0, \ldots, x_n^0)$, $F_i(x_1^1, \ldots, x_n^1) \in CL(X_i)$ and $x_i^1 \in F_i(x_1^0, \ldots, x_n^0)$, $i = 1, \ldots, n$, there exist points $x_i^2 \in F_i(x_1^1, \ldots, x_n^1)$, $i = 1, \ldots, n$, such that

$$d_i(x_i^1, x_i^2) \leqslant H_i\left[F_i(x_1^0, \ldots, x_n^0), F_i(x_1^1, \ldots, x_n^1)\right] + \nu. \tag{11}$$

By induction, we obtain the sequences $\{x_i^k\}_{k=1}^{\infty}$, $i = 1, \ldots, n$, of points of X_i, $i = 1, \ldots, n$, such that $x_i^k \in F_i(x_1^{k-1}, \ldots, x_n^{k-1})$, $i = 1, \ldots, n$, $k = 1, 2, \ldots,$ and

$$\begin{aligned} d_i(x_i^k, x_i^{k+1}) \leqslant{}& H_i\left[F_i(x_i^{k-1}, \ldots, x_n^{k-1}), F_i(x_1^k, \ldots, x_n^k)\right] \\ &+ \frac{\nu^k}{(1-c)^{k-1}}, \qquad i = 1, \ldots, n, \quad k = 1, 2, \cdots. \end{aligned} \tag{12}$$

From (11), (8), (10) and (9) we obtain

$$\begin{aligned} d_i(x_i^1, x_i^2) &\leqslant \sum_{k=1}^{n} a_{i,k} d_k(x_k^0, x_k^1) + \sum_{k=1}^{n} b_{i,k} D_k\left[x_k^0, F_k(x_1^0, \ldots, x_n^0)\right] \\ &\quad + cD_i\left[x_i^1, F_i(x_1^1, \ldots, x_n^1)\right] + \nu \\ &\leqslant \sum_{k=1}^{n} (a_{i,k} + b_{i,k}) d_k(x_k^0, x_k^1) + cd_i(x_i^1, x_i^2) + \nu \\ &\leqslant \sum_{k=1}^{n} c_{i,k} r_k + cd_i(x_i^1, x_i^2) + \nu \\ &\leqslant \nu r_i + cd_i(x_i^1, x_i^2) + \nu. \end{aligned}$$

Thus

$$d_i(x_i^1, x_i^2) \leqslant \frac{\nu}{1-c} r_i + \frac{\nu}{1-c} \leqslant 2\frac{\nu}{1-c} r_i.$$

Recalling (12), (8), (10), (9) and the induction principle, we obtain

$$d_i(x_i^k, x_i^{k+1}) \leqslant (k+1)(\nu/(1-c))^k r_i, \qquad i = 1, \ldots, n, \quad k = 1, 2, \cdots.$$

Now we have

$$\begin{aligned} d_i(x_i^k, x_i^{k+m}) &\leqslant d_i(x_i^k, x_i^{k+1}) + \cdots + d_i(x_i^{k+m-1}, x_i^{k+m}) \\ &\leqslant (k+1)\left(\frac{\nu}{1-c}\right)^k r_i + \cdots + (k+m)\left(\frac{\nu}{1-c}\right)^{k+m-1} r_i \\ &\leqslant \sum_{s=k}^{\infty} (s+1)\alpha^s r_i \leqslant (k+2)\alpha^k (1-\alpha)^{-2} r_i \end{aligned}$$

for all $k, m \geqslant 1$, $i = 1, \ldots, n$, where $\alpha = \nu/(1-c)$. Thus $\{x_i^k\}_{k=1}^{\infty}$, $i = 1, \ldots, n$, are Cauchy sequences and, therefore, $x_i^k \to u_i \in X_i$, $i = 1, \ldots, n$. We claim that $(u_1, \ldots, u_n)$ is a fixed point of $(F_1, \ldots, F_n)$. Actually

$$\begin{aligned} D_i\left[u_i, F_i(u_1, \ldots, u_n)\right] &\leqslant d_i(u_i, x_i^{k+1}) + D_i\left[x_i^{k+1}, F_i(u_1, \ldots, u_n)\right] \\ &\leqslant d_i(u_i, x_i^{k+1}) + H_i\left[F_i(x_1^k, \ldots, x_n^k), F_i(u_1, \ldots, u_n)\right] \\ &\leqslant d_i(u_i, x_i^{k+1}) + \sum_{s=1}^{n} a_{i,s} d_s(x_s^k, u_s) \end{aligned}$$

$$+ \sum_{s=1}^{n} b_{i,s} D_s\left[x_s^k, F_s\left(x_1^k, \ldots, x_n^k\right)\right] + cD_i\left[u_i, F_i(u_1, \ldots, u_n)\right]$$

$$\leqslant d_i\left(u_i, x_i^{k+1}\right) + \sum_{s=1}^{n} a_{i,s} d_s\left(x_s^k, u_s\right)$$

$$+ \sum_{s=1}^{n} b_{i,s} d_s\left(x_s^k, x_s^{k+1}\right) + cD_i\left[u_i, F_i(u_1, \ldots, u_n)\right].$$

Hence,

$$D_i\left[u_i, F_i(u_1, \ldots, u_n)\right]$$

$$\leqslant \frac{1}{1-c}\left[d_i\left(u_i, x_i^{k+1}\right) + \sum_{s=1}^{n} a_{i,s} d_s\left(x_s^k, u_s\right)\right.$$

$$\left. + \sum_{s=1}^{n} b_{i,s} d_s\left(x_s^k, x_s^{k+1}\right)\right] \to 0 \quad \text{as } k \to \infty.$$

Since $F_i(u_1, \ldots, u_n)$ is closed, this means that $u_i \in F_i(u_1, \ldots, u_n)$, $i = 1, \ldots, n$, which completes the proof.

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